# MATHEMATICS OF STATISTICS

## PART ONE

BY

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## PREFACE TO THE SECOND EDITION

Striking examples of the utility and scope of the science of statistics have occurred in recent years. As Professor Harold Hotelling remarks (*Annals of Mathematical Statistics*, vol. 11, 1940, pp. 457–470):

> Indeed it seems as if the exploitation of the business and manufacturing possibilities of statistical methods has only begun and that limitless further fields are coming into view.

The widespread use of statistical methods and the gratifying interest shown in the present book have made possible a second edition at this time. This opportunity has been used to polish and clarify certain portions of the text. For suggestions leading to the excision of obscurities I am indebted to many of my students and to a number of friends in other universities, particularly to Professors Irving W. Burr, John H. Curtiss, Henry Scheffé, Guy G. Speeker, and Howard E. Wahlert. Of course, full responsibility for any remaining errors or other defects is my own.

J. F. K.

## PREFACE TO THE FIRST EDITION

The field of statistics is many sided and ranges over different levels. However, between the levels of clerical work at one extreme and mathematical research at the other extreme, there is a well-defined methodology, mathematical in nature, which underlies the specialized applications in the departments of economics, psychology, education, and biology.

This book is an elementary text dealing with the mathematics of statistics. Fortunately, a considerable part of the descriptive methodology of statistics can be understood by those having relatively little knowledge of college mathematics. Although no mathematics beyond the ordinary Freshman course in college algebra is required for a profitable reading of this text, a certain degree of mathematical maturity and intelligence is presupposed. To achieve the maximum success perhaps only the best of those students whose mathematical preparation is limited to the minimum prerequisite should be encouraged to study it. Occasionally, material is introduced to sharpen the interest and challenge the ability of the more advanced student without interrupting the main developments or discouraging those less mature.

In writing this book, considerable selection of material necessarily had to be made. The omission of certain topics will be noted in the table of contents. Judging from my own experience, and that of others, the theory of sampling cannot be taught satisfactorily at the level for which Part I is intended. At best only a superficial use of formulas could be hoped for. Consequently, I have elected to defer this subject to Part II where a systematic treatment can be given. With regard to time series analysis, Professor J. Neyman says in his *Lectures And Conferences On Mathematical Statistics* (p. 106),

> We start by trying to split each of the series into several parts, which we arbitrarily assume to be additive. One of these parts is the trend, which we estimate perhaps by fitting a low order parabola to the whole series available. The next part is the "business cycle." The third part is the "seasonal variation," which we frequently estimate by calculating moving averages. Finally, the remainder is considered to arise from random causes, and we concentrate on the question whether such a remainder in one of the variables is correlated

with that in some other. All this procedure seems to me very artificial and arbitrary. . . . In my opinion the whole problem of time series must be treated from a point of view that is quite different from the traditional one just described.

I concur in this opinion and I believe that no useful purpose would be served by drilling students in the traditional procedures.

Throughout the book the student is encouraged and stimulated to master fundamental principles and concepts. Essentially, the job of every statistician is to take hold of situations and disentangle them by the techniques of the science. Therefore, considerable emphasis is placed on technique. I have tried to develop in the student the ability to use symbolism creatively as a language. Numerous examples are given to clarify concepts and illustrate processes. Over two hundred exercises are included. It is intended that these exercises should be handled as in a mathematics course. No laboratory, so-called, is necessary.

Nowadays, no little importance is attached to motivation. I have constantly held in mind the necessity of making the subject interesting and stimulating to the beginning student. Nevertheless, I venture the opinion that the best motivation for intelligent students is the feeling that their teacher knows his subject.

In preparing the manuscript a large number of books and papers have been examined and perhaps leaned upon. No claim to originality is made except possibly in the matter of arrangement and pedagogical approach. Numerous references to the scholarly achievements of others are cited. It is hoped that the serious student will read some of these and thereby widen his perspective and enhance his interest.

In conclusion, I wish to express my deep appreciation to Professor Allen T. Craig and Dr. Mason E. Wescott who critically read the manuscript and made many suggestions for its improvement.

April, 1939 JOHN F. KENNEY

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# MATHEMATICS OF STATISTICS

## INTRODUCTION

**1. Definition.** The word *statistics* is used in at least two different senses. Construed as plural it refers to the systematic presentation of quantitative data. Used in a singular sense, the word *statistics* refers to the science which has for its object the classification and analysis of quantitative data so that intelligent judgments may be passed upon them.

It is usually clear from the context which meaning[1] is intended, although some persons prefer the expression *statistical methods* for this second meaning. Statistical methods are all those devices used in the collection and analysis of data. The *theory of statistics* is the exposition of statistical methods and is of a mathematical nature.

**2. Scope.** There used to be a widespread misapprehension that statistics is a branch of economics. As a matter of fact, statistical problems arise in many different fields — biology, economics, engineering, insurance, education, physics, and astronomy, as well as various branches of business. The exploration of certain aspects of nearly every field involves some phase of statistical theory. Indeed, certain types of statistical methodology may have almost unexpected applications — the discovery, for example, that the life of physical property[2] is governed by much the same statistical rules as govern the lives of human beings, and hence, that life tables may be applied to both. Physicists have discovered that many of the problems in the modern theory of the structure of the atom are essentially statistical in nature. In recent years industrial companies have placed an increasing reliance on statistical methods in controlling the quality of goods during manufacture.

Statistics as a science is making contributions to all the sciences. On the other hand, some sciences like biometry and physics have

[1] In addition to the two meanings given above, another has crept into the recent literature where reference is made to a *statistic*. This term will be explained later.

[2] *Life Expectancy of Physical Property* — E. B. Kurtz. Ronald Press.

contributed much in the development of statistics and its terminology. The following quotation from *Science* may appropriately be mentioned here:

> The extension of the scope of quantitative methods through the medium of statistical analysis is one of the most significant things going on in the scientific world at the present time.[1]

The importance of statistical method in present-day thinking has been well stated, as follows:

> More and more the modern temper relies upon statistical method in its attempts to understand and to chart the workings of the world in which we live. Particularly in those sciences which deal with human beings, whether in their physical and biological aspects or in their social, economic, and psychological relations, the spirit of our time asks that its conclusion be based not so much upon the distinctive reactions of one or two individuals as upon the observation of large numbers of individuals, the measurement of their common likenesses and the extent of their diversity. As the data thus gathered from mass phenomena become extensive, it becomes imperative to have methods of organization to bring the facts within the compass of our understanding, methods of analysis to make the essential relations appear out of the mass of detail in which they are hidden, and methods of classification and description to facilitate the presentation of the data for the study and consideration of other persons. Thus statistical method becomes a telescope through which we can study a larger terrain than would be accessible to our unaided vision.[2]

**3. Statistical Methods in the Social Sciences.** Because statistics is fundamentally the study of aggregates of individuals, rather than of individuals, whether these *individuals* be observations or measurements or persons, it is apparent that statistical methods are essential to social studies. Indeed it has been said that it is principally by the aid of such methods that these studies may be raised to the rank of sciences.

This particular dependence of social studies upon statistical methods is mentioned in a recent book[3] from which we quote the following:

> If, as seems probable, our present uncoördinated large-scale business is to be further developed into an efficiently managed instrument of production serving the needs of the people, then statistics, together with mathematical economics, will emerge among the most important tools of the social sciences. For it is by

[1] *Science.* January 18, 1929.

[2] *Mathematics and Statistics* — Walker. Sixth Yearbook, National Council of Teachers of Mathematics.

[3] Reprinted by permission from *Methods of Statistical Analysis* by Davies and Crowder, published by John Wiley and Sons, Inc.

means of averages, dispersions, coefficients of variability, trends, and regressions, as pictured in control charts, that management is able to visualize and direct the movements of large masses of population.

The work of the statistician is much like that of the map maker who presents the traveler with a sketch of important highways, showing the locations of towns and geographical features. The map is not a picture of reality. It shows cities as dots, and rivers as lines. It has purposely omitted the interesting details of scenery and the still more important features of human interest which lie along the route and which constitute the traveler's real objectives. Nevertheless, as a means of reaching these objectives, the map is extremely useful. And so it is with statistics in the hands of the business executive and statesman. Back of the charts are human beings with their varying characteristics and vital interests, few of which can be described in figures. Yet as a means of serving these interests, of keeping trade moving from one region to another, of allocating investment and labor, and of apportioning relief to maladjusted industries and dependent classes, statistics and mathematical methods are important, and are becoming increasingly important with the growing complexity of society.

It may be said that the study of statistics is not merely an attempt to describe what actually occurs, though it must begin at this point, but in its broader aspects it is the logical background of business and social management. Hence what appears now to be mere abstraction may later become the basic necessity of an applied science. Eventually, it may be assumed, the social arts of business and politics will rest upon as substantial a theoretical and mathematical background as physics, chemistry, and engineering.

**4. Mathematics and Statistics.** Statistical problems are of interest, therefore, not only to the worker in the particular field but also to the mathematician, inasmuch as methods adequate to the treatment of these problems can best be presented in the precise and accurate language of mathematics. Moreover, statistical methods are grounded in statistical theory which is a branch of applied mathematics.

Although it is true that some statistical problems are ultimately problems in advanced mathematics, many of which mathematicians have not yet been able to solve, nevertheless a large and interesting part of statistical analysis requires mathematics no more advanced than elementary algebra.

It has been said that sooner or later every true science tends to become mathematical. The notation of mathematics is simply a language and it is not limited to any particular field of knowledge. The following quotations are inserted to help the student approach the study of statistics in the proper spirit.

1. Mathematics, the science of the ideal, becomes the means of investigating, understanding, and making known the world of the real. — White.

2. Probably among all the pursuits of the university, mathematics preeminently demands self-denial, patience, and perseverance. . . . — Todhunter.

3. From time immemorial, there has been but one way to become a mathematician and there will never be another: it is a way interior to the subject and involves years of assiduous toil. Short-cuts to mathematical scholarship there are none, whether the seeker be a philosopher or a king. — Keyser.

4. Will is the creative force. Without the will to learn there is no learning. And when the will is feeble and confused, learning lags. — Mursell.

5. The theory of statistics is not easy, not so much because it is abstruse, as because the ideas are new to most people, and a good deal of hard thinking and patient work will be necessary. . . . Statistical work always involves a lot of computing [and] there is no better way of learning statistics than by working through examples. — Tippett.

**5. Problem Assignments.** The student should realize at the outset that statistical methods are not substitutes for thinking but are aids and supplements to it. A superficial knowledge of statistical technique cannot take the place of good judgment. Mere ability to substitute in formulas should not be confused with genuine statistical sophistication and insight. To the serious and capable student who intends to master this course, formulas will be a set of functioning concepts and tools rather than machines into which material may be fed to grind out a meaningless answer.

This opportunity is also taken to point out that even mathematical discourse consists of sentences. Punctuation should not be omitted in sequences of equations and other mathematical statements. (It is admitted, however, that many of us find this difficult to remember.)

Throughout the book exercises are inserted to give the student an opportunity to test his knowledge of the theory and methodology, and to develop his power of analysis. In grading the solutions, value will be attached to accuracy, thoroughness, neatness, and systematic arrangement of the work.

**6. Calculating Machines.**[1] A full description of the parts of a calculating machine and their operation may be obtained from an *Instruction Book* which is furnished by the manufacturer, so only a brief description will be given here.

A calculating machine is constructed to add and substract. By means of continued addition or subtraction, operations involving multiplication, division, and square root can also be performed with great speed.

[1] The early history of modern computing machines is outlined in the *American Mathematical Monthly*, vol. 31 (1924), pp. 422–429.

In addition to a keyboard on which numbers can be punched, most machines have a sliding carriage, carrying two dials one above the other. These dials are called *revolution register* (upper dial) and *product register* (lower dial). In finding a product $nx$, one of the factors $n$ is punched on the keyboard and as the motive crank at the side is turned,[1] the other factor $x$ appears on the upper dial. The product $nx$ is then read from the lower dial.

An important property of the modern calculating machine is its adaptability to short cuts and combinations of operations. For example, one may multiply two numbers $nx$ together and add the result to a third number $k$ without tabulating the intermediate steps. This is accomplished by punching the number $k$ on the keyboard, transferring it to the lower dial (product register), and then proceeding as in finding the product $nx$. The result $nx + k$ is then read from the lower dial. An extension of this procedure is especially useful in a series of computations where $k$ and $n$ are constant and various values are assigned to $x$. To describe the procedure, suppose it is required to calculate the successive values of $12 + 6x$ for $x = 5, 7, 15, 12$, etc. The number $k = 12$ is first registered on the lower dial, then the factor $n = 6$ is placed on the keyboard, and by turning the crank forward five times to make the first value of $x = 5$ appear on the upper dial, the result $12 + 6 \times 5$ appears on the lower dial. Instead of clearing the dial, the crank is now turned forward twice more to rebuild the value $x = 5$ into $x = 7$, and the result $12 + 6 \times 7$ can be read from the lower dial. In rebuilding $x = 15$ into $x = 12$ the crank is turned backwards. This procedure can be repeated until all the required values of $12 + 6x$ have been calculated. A process of this sort is called the *continuous method* of calculating.

In most of the exercises in this course, the computations are not laborious and calculating machines are not required. However, if machines are available they may be used to advantage in Chapters IV and VI. The student who desires to develop skill on a calculating machine should begin now to study an *Instruction Book* and practice the fundamental operations explained there.

**7. Collateral Reading.** Perhaps no single textbook can meet all the needs of all students of statistics. There are several good books on elementary statistics which, although not fundamentally different,

[1] The beginner will probably wish to practice on a manually operated machine before attempting to use the high-speed electric and automatic machines.

present different points of view on certain topics and treat them with varying degrees of emphasis depending upon the field of major interest. At least some of the books listed below should be readily available on the reserve shelf of the library. The list should be useful to those who wish to study more fully certain details in which they may be interested.

1. Bivins — *The Ratio Chart in Business.* Codex Book Co.
2. Burgess — *The Mathematics of Statistics.* Houghton Mifflin and Co.
3. Camp — *The Mathematical Part of Elementary Statistics.* D. C. Heath and Co.
4. Deming — *Statistical Adjustment of Data.* John Wiley & Sons, Inc.
5. Freeman — *Industrial Statistics.* Wiley.
6. Garrett — *Statistics in Psychology and Education.* Longmans, Green and Co.
7. Glover — *Tables of Applied Mathematics.* Wahr.
8. Haskell — *Graphic Charts in Business.* Codex Book Co.
9. Mills — *Statistical Methods, Revised.* Henry Holt and Co.
10. Pearl — *Medical Biometry and Statistics.* W. B. Saunders and Co.
11. Rider — *Statistical Methods.* Wiley.
12. Scarborough — *Numerical Mathematical Analysis.* The Johns Hopkins Press.
13. Snedecor — *Statistical Methods.* Collegiate Press, Inc., Ames, Iowa.
14. Treloar — *Statistical Reasoning.* Wiley.
15. Walker — *Elementary Statistical Methods.* Holt.
16. Yule and Kendall — *The Theory of Statistics.* Griffin and Co.

# CHAPTER I

## FREQUENCY DISTRIBUTIONS

**1. Variables and Constants.** A *variable* is a number symbol which may take on any value in a set of values which is called its *range*. A *constant* is a symbol whose range consists of only one value (in a particular discussion or situation). Letters toward the end of the alphabet, such as $x$, $y$, $u$, and $v$, are commonly used to denote variables. When a constant does not have a definite value such as 2, $\frac{1}{3}$, $\pi$, and so forth, it is customary to represent the constant by a letter toward the beginning of the alphabet.

Two famous constants are

$$\pi = 3.14159\ldots, \quad e = 2.71828\ldots$$

They occur in mathematics in many important, interesting, and even curious ways. As instances of the latter, the following examples are noteworthy.

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots, \quad \text{where } n! = n(n-1)(n-2)\cdots 1.$$

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}}$$

The expression for $e$ is called a convergent infinite series and that for $\pi/4$ a continued fraction.

**2. Variates.** In general, statistical data are obtained by taking observations or measurements on one or more variables. The values thus obtained are sometimes called *variates*.[1] For example, in computing the average monthly rainfall of a region the variable is rainfall and the amount of rainfall for any month is a variate. Like-

[1] A somewhat different usage of this term is explained in Part II.

wise, if the bank clearings of the city of Madison are under consideration, then the variable is bank clearings, and the clearings for any specified interval are variates. If we denote a variable by $x$ then the $N$ values which it takes on are denoted by $x_1, x_2, \cdots, x_N$.

Variates are of two kinds: *continuous* and *discrete*. Continuous variates are values of a variable which, theoretically, can be measured to any degree of fineness, such as heights, weights, temperatures, ages. All the numbers between $x = 0$ and $x = 1$ form a set of continuous variates. But if we restrict $x$ to the rational numbers in this interval we have a set of separate and distinct values with "vacant" spaces between them. Values of a variable which are thus restricted to particular values in order to have any meaning are called discrete variates. Other examples of discrete variates are: size of families, closing prices of stocks, " successes " in tossing a coin. A set of discrete variates is usually obtained by counting whereas continuous variates are usually obtained by measurement.

**3. Accuracy of Measurements.** In the case of continuous variates, the observed values as recorded can never be absolutely established by measurement. Thus, the height or weight of an object can be measured only approximately, the error depending upon the precision of the instrument and the care and accuracy of the observer. However, it is not always necessary that measurements be recorded as accurately as it is possible to make them. Similarly, in the case of discrete variates the standard of accuracy used may be less than it is possible to obtain. In population statistics, for example, it may be sufficient to record the numbers to the nearest thousand, with three zeros at the end to fill out to the decimal point. Thus,

| *City* | *Population* |
|---|---|
| A | 326,000 |
| B | 729,000 |

On the other hand, the exact number of students in a university might be required. The degree of accuracy needed is determined by the purpose of the investigation and it is limited by the closeness with which the variables can be measured.

It follows, therefore, that the degree of accuracy in the final result of a problem involving computations is limited by that of the original data. Students sometimes carry results of problems to five or more decimal places when the original data do not justify more than two

or three decimal places. A table of measurements which constitutes the raw data for a statistical investigation should always specify the degree of accuracy in the readings. Thus, if monthly rainfall is being measured to the nearest hundredth of an inch, and one measurement seems to be exactly 5 inches, it should be recorded as 5.00 inches, with two zeros. A measurement that is merely recorded as 5 means it is correct to the nearest integer and its true value lies between 4.5 and 5.5, whereas 5.00 means the true value is known to lie between 4.995 and 5.005. The three digits in 5.00 are said to be significant.

**4. Necessity for Classification.** After the data have been collected in any statistical investigation the first step has to do with introducing order in the raw material. Usually we have some hundreds of variates which have been recorded merely in the arbitrary order in which the observations or measurements happened to be made. But in order to analyze a series of variates so that intelligent judgments may be formed about it or that comparisons may be made between two series of variates, proper classification is necessary and of prime importance.

Such classification is not always an easy thing to effect, because it is the one part of statistical methods for which no very definite rules can be given. Most people, until they have tried, imagine that to collect and arrange data in classes and in tables is a straightforward procedure involving no great technique or experience. Although much can be learned from a careful study of the illustrations and discussions that appear in the following pages and the compilations of reputable bureaus such as the census volumes, nevertheless, experience is the best teacher in effecting the most appropriate classification for any set of variates.

**5. Tabulation.** In carrying out the process of classification, it becomes natural to arrange the results in tabular form, setting forth clearly and explicitly the statistics one wishes to present. In drawing up any table the following general rules should be observed:

(1) Every table must be self-explanatory. To accomplish this the title should be short, but not at the expense of clearness.
(2) Full explanatory notes, when necessary, should be incorporated in the table, either directly under the descriptive title and before the body of the table, or else directly under the form.
(3) The columns and rows should be arranged in a logical order to facilitate comparisons.

(4) In tabulating long columns of figures, spaces should be left after every five or ten rows. Long unbroken columns are confusing, especially when one is comparing two numbers in a row but in widely separated columns.

(5) If the numbers tabulated have more than three significant figures, the digits should be grouped in threes. Thus, one should write 4 685 732, not 4685732.

(6) Double lines at the top (or at the top and bottom) may enhance the effectiveness of a table. If the table nicely fills the width of the page, no side lines should be used. In such cases the omission of the side lines will have the tendency to emphasize the other vertical lines and cause the interior columns to stand out better. The columns should not be widely separated and the form of a narrow, compact table should have its side lines.

The following points are particularly important in practical work:

(7) Source of data should be included.

(8) Units of the data presented should be clear.

(9) Accuracy of transcription must not only be striven for but actually achieved. A reader who finds one error (even though this be the only one) is likely to disparage the whole table.

TABLE 1 — GRADES OF 100 STUDENTS IN FRESHMAN MATHEMATICS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 75 | 86 | 66 | 86 | 50 | 78 | 66 | 79 | 68 | 60 |
| 80 | 83 | 87 | 79 | 80 | 77 | 81 | 92 | 57 | 52 |
| 58 | 82 | 73 | 95 | 66 | 60 | 84 | 80 | 79 | 63 |
| 80 | 88 | 58 | 84 | 96 | 87 | 72 | 65 | 79 | 80 |
| 86 | 68 | 76 | 41 | 80 | 40 | 63 | 90 | 83 | 94 |
| 76 | 66 | 74 | 76 | 68 | 82 | 59 | 75 | 35 | 34 |
| 65 | 63 | 85 | 87 | 79 | 77 | 76 | 74 | 76 | 78 |
| 75 | 60 | 96 | 74 | 73 | 87 | 52 | 98 | 88 | 64 |
| 76 | 69 | 60 | 74 | 72 | 76 | 57 | 64 | 67 | 58 |
| 72 | 80 | 72 | 56 | 73 | 82 | 78 | 45 | 75 | 56 |

**6. Frequency Distribution.** From the standpoint of a mathematical analysis of statistics, the most important form of tabulation is the so-called frequency distribution. Rough data do not present any clear ideas of description unless they are organized and condensed in a systematic way. We therefore partition the raw data into *classes* of appropriate size, showing the corresponding frequency of variates in each class. When any set of statistics is systematically

arranged in this way it is called a frequency distribution. For example, upon an examination of the raw data of Table 1, it is difficult to state any very definite conclusions as to whether these grades represent preponderantly good students or poor ones. The frequency distribution of Table 2, however, does give us more precise infor-

TABLE 2 — FREQUENCY TABLE OF 100 GRADES

| Class Limits | Tally Marks | Frequency |
|---|---|---|
| 30-39 | // | 2 |
| 40-49 | /// | 3 |
| 50-59 | 𝍸 𝍸 / | |
| 60-69 | 𝍸 𝍸 𝍸 𝍸 | |
| 70-79 | 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 // | |
| 80-89 | 𝍸 𝍸 𝍸 𝍸 𝍸 | 25 |
| 90-99 | 𝍸 // | 7 |
| Total | | 100 |

mation. We see at a glance that there were 32 students with grades between 70 and 80, and that all but 16 had grades of 60 or above. In Table 3, the confusion of detail is still more apparent. The corresponding frequency distribution is given in Table 4.

The width of a class is called the class interval, and in general the successive class intervals should be of equal width. The mid-value of such an interval is variously called the class mark, mid-value, central value. The width of a class interval is therefore seen to be the common difference between two consecutive class marks. It is also the difference between the lower (or upper) limit of two successive classes. Thus, in Table 4, the class interval is half an inch and the successive class marks are 0.245, 0.745, etc., inches.

**7. Class Intervals.** Grouping variates into the most appropriate number of classes is a matter of judgment. The choice of intervals to be used in tabulating any particular set of variates depends upon the nature and characteristics of the data and the purpose for which it is to be used. In the case of discrete variates, the unit is a natural interval and sometimes it is satisfactory. (See Tables 10 and 11.) However, for both discrete and continuous variates the following conditions should guide the choice: (a) We desire to be able to treat all the values assigned to any one class, without serious error, as if

TABLE 3 -- MONTHLY RAINFALL AT IOWA CITY, 1890–1925

| *Year* | *Jan.* | *Feb.* | *Mar.* | *Apr.* | *May* | *June* | *July* | *Aug.* | *Sept.* | *Oct.* | *Nov.* | *Dec.* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1890 | 2.75 | 0 75 | 1 80 | 1.83 | 2 20 | 7 99 | 0.30 | 2.29 | 1.44 | 2.11 | 1.56 | 0.31 |
| 1891 | 1.49 | 1 30 | 4 41 | 1.11 | 4 46 | 2.80 | 3.01 | 3.45 | 2.33 | 1.63 | 2.93 | 2.72 |
| 1892 | 1 46 | 1.23 | 3 15 | 4 30 | 9.23 | 8.29 | 6.20 | 2 50 | 1.18 | 1.02 | 1.38 | 2.84 |
| 1893 | 1 18 | 1.75 | 2.82 | 4 37 | 1.79 | 3 01 | 3.56 | 1.64 | 3.07 | 1.98 | 1.75 | 1.52 |
| 1894 | 1.95 | 1 64 | 2 03 | 2 72 | 3.09 | 2 40 | 0.90 | 2 40 | 4 96 | 2 30 | 1 80 | 0.98 |
| 1895 | 2 37 | 0.64 | 1 25 | 1 66 | 4 26 | 1 10 | 10.10 | 1.77 | 3.43 | 1.38 | 1.78 | 2.84 |
| 1896 | 0 70 | 1 51 | 0 92 | 5 11 | 4 10 | 1 86 | 7.04 | 2.44 | 1 82 | 2 74 | 1 16 | 0.55 |
| 1897 | 3.66 | 1 30 | 2.07 | 4 60 | 3 11 | 2.38 | 3 83 | 1.85 | 3.54 | 0.33 | 1.98 | 2.48 |
| 1898 | 4.62 | 1.15 | 3.02 | 2 89 | 4 80 | 3 26 | 2 27 | 2 85 | 2 54 | 4 38 | 1 10 | 0.53 |
| 1899 | 0 59 | 1 82 | 1.43 | 3 23 | 9 49 | 4 50 | 3 78 | 2 39 | 0.93 | 1 66 | 1.15 | 1.93 |
| 1900 | 0 73 | 2.20 | 3.32 | 3 31 | 4.31 | 2.18 | 5 25 | 6 27 | 4 35 | 3 61 | 1 43 | 0.75 |
| 1901 | 1 07 | 1.97 | 3.62 | 2 36 | 1 54 | 3 33 | 1 29 | 0 66 | 2 56 | 1.78 | 0.79 | 2.34 |
| 1902 | 1 29 | 0.85 | 1 29 | 1 91 | 3 75 | 7 46 | 6 89 | 10 91 | 5 87 | 3.12 | 2.25 | 2.21 |
| 1903 | 0.67 | 1.03 | 1 86 | 3 11 | 6 90 | 1.95 | 4 76 | 3 45 | 5.38 | 3 60 | 0 97 | 1 27 |
| 1904 | 1 74 | 0.84 | 2.73 | 5 49 | 2.68 | 2 14 | 2.49 | 3 93 | 3 12 | 1 59 | 0 25 | 1 96 |
| 1905 | 1 22 | 1 90 | 2 28 | 3 36 | 5 37 | 6 68 | 3 59 | 2 62 | 1.54 | 5 36 | 2.92 | 1.04 |
| 1906 | 2 51 | 1.73 | 2 25 | 1 83 | 2.33 | 3.64 | 1.42 | 5 34 | 0 89 | 1 48 | 3 08 | 1.64 |
| 1907 | 2 12 | 0 22 | 1.59 | 1.58 | 5 47 | 6 04 | 9 21 | 2 98 | 2 85 | 0 86 | 1.07 | 0 53 |
| 1908 | 0.32 | 2.08 | 2 94 | 2 78 | 7.78 | 2.87 | 5.40 | 7 47 | 1.82 | 1.99 | 1 84 | 0.43 |
| 1909 | 1 97 | 1.09 | 2 00 | 7.21 | 4.40 | 4 58 | 5 75 | 1 88 | 2.43 | 1.59 | 4 88 | 2.52 |
| 1910 | 1 79 | 0.39 | 0.28 | 2.56 | 3 57 | 0.98 | 2.22 | 4.98 | 3.87 | 0 57 | 0.69 | 0.46 |
| 1911 | 0 87 | 4 82 | 1.30 | 3 02 | 4 74 | 2 98 | 3.70 | 4.27 | 5.07 | 2.78 | 3.01 | 2 29 |
| 1912 | 0 26 | 1.21 | 2.30 | 3 50 | 2 88 | 2.60 | 3.60 | 3 62 | 2.67 | 3 54 | 1.11 | 0.75 |
| 1913 | 1 19 | 1.42 | 2 69 | 1.83 | 6.91 | 6 28 | 0.39 | 2.97 | 3.19 | 3.66 | 0.46 | 1.02 |
| 1914 | 1.28 | 0 93 | 2 63 | 2.37 | 4 87 | 5.32 | 1.53 | 2.99 | 7.97 | 1.65 | 0.37 | 1.89 |
| 1915 | 2.15 | 2.42 | 0.92 | 0.65 | 7.65 | 4.33 | 8.11 | 1.80 | 9.31 | 1.84 | 1.80 | 0.80 |
| 1916 | 3.18 | 0.59 | 5.06 | 1.83 | 5.99 | 3 92 | 1.57 | 2.83 | 3.49 | 3.19 | 1.42 | 1.15 |
| 1917 | 1.09 | 0.19 | 2.19 | 3.43 | 7.33 | 6.49 | 2.84 | 2.79 | 6.23 | 2.28 | 0.30 | 0.57 |
| 1918 | 1.10 | 1 46 | 0.33 | 3.43 | 6.22 | 8.36 | 4.87 | 6.72 | 2.00 | 2.05 | 2.10 | 1.62 |
| 1919 | 0.08 | 2.63 | 2.65 | 4.28 | 4.49 | 7.07 | 1.03 | 2.67 | 5.10 | 4.01 | 3.84 | 0.61 |
| 1920 | 0.84 | 1.33 | 4.22 | 4 75 | 3 76 | 2.86 | 2.79 | 2.90 | 1.20 | 0.98 | 1.80 | 2.45 |
| 1921 | 0.35 | 0.49 | 2.46 | 6.20 | 4.44 | 2.46 | 3.59 | 8.61 | 7.83 | 2.47 | 0.74 | 3.19 |
| 1922 | 1.11 | 1.46 | 2.18 | 3.49 | 5.52 | 0 28 | 6.46 | 1.03 | 2.91 | 1.06 | 5.28 | 0.49 |
| 1923 | 1.09 | 0.67 | 4.83 | 0.86 | 2.63 | 6.21 | 2.37 | 4.01 | 9.27 | 2.35 | 1.13 | 0.73 |
| 1924 | 1.35 | 0.83 | 2.10 | 1.09 | 1.69 | 8.71 | 3.67 | 5.67 | 2.60 | 1.64 | 0.93 | 1.75 |
| 1925 | 0.29 | 1.04 | 0.99 | 3 07 | 1.06 | 5.61 | 3.63 | 3.14 | 5.59 | 3.90 | 1.00 | 1.66 |

they were equal to the class mark for that interval; e.g., as if all 23 items in the first class of Table 4 were exactly 0.245 inches, etc. (*b*) For convenience and brevity we desire to make the interval as large as possible subject to the first condition. These conditions will generally be fulfilled if the interval is so chosen that the whole num-

TABLE 4 — FREQUENCY TABLE OF MONTHLY RAINFALL AT IOWA CITY, 1890–1925

| *Class Interval* | *Mid-x* | *Frequency* |
|---|---|---|
| 0.00– 0.49 | 0.245 | 23 |
| 0.50– 0.99 | 0.745 | 42 |
| 1 00– 1.49 | 1.245 | 58 |
| 1 50– 1.99 | 1.745 | 62 |
| 2.00– 2.49 | 2.245 | 49 |
| 2.50– 2.99 | 2.745 | 47 |
| 3.00– 3.49 | 3 245 | 32 |
| 3 50– 3 99 | 3 745 | 27 |
| 4.00– 4.49 | 4 245 | 18 |
| 4.50– 4.99 | 4.745 | 15 |
| 5.00– 5 49 | 5.245 | 14 |
| 5.50– 5.99 | 5 745 | 7 |
| 6.00– 6.49 | 6.245 | 10 |
| 6.50– 6 99 | 6.745 | 5 |
| 7.00– 7.49 | 7.245 | 6 |
| 7.50– 7.99 | 7.745 | 5 |
| 8.00– 8.49 | 8.245 | 3 |
| 8 50– 8.99 | 8.745 | 2 |
| 9.00– 9.49 | 9.245 | 5 |
| 9.50– 9.99 | 9 745 | 0 |
| 10.00–10.49 | 10.245 | 1 |
| 10.50–10.99 | 10.745 | 1 |
| Total | | 432 |

ber of classes lies between 10 and 25. A small number of classes may "cover up" too much detail whereas a large number may reveal too much detail for one to comprehend readily (which is just the objection to the table of original data). A preliminary inspection of the data should accordingly be made and the highest and lowest values selected. Dividing the difference between these by the tentative number of classes, we have our approximate value

TABLE 5 — MONTHLY RAINFALL AT DES MOINES, 1890–1925

| *Year* | *Jan.* | *Feb.* | *Mar.* | *Apr.* | *May* | *June* | *July* | *Aug.* | *Sept.* | *Oct.* | *Nov.* | *Dec.* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1890 | 2.62 | 1.17 | 0.91 | 0.78 | 3.00 | 4.91 | 1 10 | 3.35 | 1.57 | 4.48 | 0.74 | 0.11 |
| 1891 | 1.82 | 1 13 | 2 25 | 2 12 | 3 29 | 5 60 | 2 78 | 4.22 | 1 64 | 2 41 | 1.34 | 1.54 |
| 1892 | 1.60 | 1 35 | 2.47 | 3 36 | 8 77 | 3 41 | 8.64 | 2 45 | 1.12 | 2.54 | 0 76 | 1.95 |
| 1893 | 0.56 | 1.28 | 1 15 | 5 61 | 2.84 | 4 69 | 3 55 | 1 60 | 1.33 | 0.22 | 1.51 | 1.30 |
| 1894 | 1.09 | 1 39 | 1 78 | 1.70 | 1.41 | 1 67 | 0.29 | 1 89 | 4.46 | 2 24 | 0.99 | 1.15 |
| 1895 | 1 30 | 0 60 | 0 50 | 3.41 | 2 86 | 5.26 | 3 10 | 3.57 | 3.20 | 0.29 | 0.85 | 1.86 |
| 1896 | 0 60 | 0.79 | 1.24 | 3.47 | 6 50 | 2 69 | 8.15 | 5.49 | 3.61 | 2.69 | 1.10 | 0.85 |
| 1897 | 2.02 | 0.71 | 2.13 | 7 37 | 2 31 | 3 15 | 2 88 | 1 77 | 1 56 | 0 85 | 0.34 | 1.98 |
| 1898 | 1.59 | 0.82 | 1.35 | 2.64 | 4 22 | 6 85 | 1 86 | 1.09 | 1.91 | 3.56 | 1 87 | 0.57 |
| 1899 | 0.29 | 0.57 | 1.04 | 2.22 | 6.71 | 3 53 | 3.20 | 3.53 | 1 17 | 0.59 | 1.76 | 2.12 |
| 1900 | 0.20 | 0 50 | 3 07 | 3.82 | 4.76 | 4 89 | 5.15 | 8 02 | 3.66 | 3.08 | 0 96 | 0.35 |
| 1901 | 1.01 | 1.11 | 3.02 | 2.26 | 1 40 | 2.41 | 1.72 | 0.67 | 2 60 | 2 14 | 0.40 | 1.03 |
| 1902 | 0 91 | 0.52 | 1.15 | 1.55 | 4 69 | 7 27 | 5 95 | 7 82 | 5 03 | 3.70 | 1.65 | 1.77 |
| 1903 | 0 20 | 1 12 | 1 09 | 1 64 | 0 64 | 3 06 | 3 62 | 6.72 | 1.62 | 1 32 | 0 31 | 0.09 |
| 1904 | 1 22 | 0.22 | 1 20 | 5 48 | 3 16 | 2 08 | 6 94 | 2 60 | 1.95 | 1 50 | 0.06 | 2 02 |
| 1905 | 1.08 | 1 00 | 2.16 | 3 29 | 4 44 | 5 73 | 4 53 | 5 21 | 3 17 | 3.64 | 2.34 | 0.55 |
| 1906 | 2.07 | 0.86 | 1 84 | 2 96 | 2 21 | 3 80 | 2 67 | 4 69 | 3 24 | 1 18 | 2 29 | 1.46 |
| 1907 | 0.87 | 0 93 | 1.18 | 1.48 | 2 97 | 4 13 | 10.20 | 5 03 | 2 40 | 1.70 | 1.12 | 1.01 |
| 1908 | 0.46 | 1.15 | 1.43 | 2.69 | 9 89 | 5 93 | 1 56 | 6 54 | 0.94 | 3 68 | 0.95 | 0.31 |
| 1909 | 1.61 | 0.90 | 1.56 | 5 14 | 4 24 | 7.01 | 4 41 | 0 14 | 2 06 | 2.89 | 3.71 | 2.32 |
| 1910 | 1.72 | 0.20 | 0.33 | 1.13 | 3 26 | 3.11 | 0.86 | 2.40 | 3.82 | 0.68 | 0.53 | 0.20 |
| 1911 | 0.84 | 2.91 | 1.14 | 4.23 | 2.44 | 0.75 | 1.16 | 1.82 | 7.68 | 2.61 | 1.22 | 3 18 |
| 1912 | 0.53 | 1.86 | 2.87 | 2 75 | 5 62 | 2 60 | 3 07 | 3 52 | 4.20 | 3.75 | 1.11 | 0.30 |
| 1913 | 1.10 | 0 65 | 3.03 | 3.41 | 5 06 | 3 52 | 1.05 | 3 44 | 2.65 | 2.67 | 1.03 | 1.05 |
| 1914 | 0.85 | 1 24 | 1.18 | 1.52 | 4.83 | 3 89 | 1 22 | 1.77 | 4.81 | 3.57 | 0.35 | 1.28 |
| 1915 | 1.96 | 3.20 | 1.16 | 1.36 | 8.21 | 3.60 | 9.39 | 1.71 | 4.51 | 0.43 | 1.24 | 0.65 |
| 1916 | 2.66 | 0.61 | 0.60 | 2.44 | 3.87 | 2 42 | 1.50 | 2.62 | 1.72 | 2.11 | 1.46 | 0.65 |
| 1917 | 6.53 | 0.52 | 2.30 | 5 52 | 3.94 | 8.16 | 1.58 | 1.82 | 1.99 | 0.92 | 0.21 | 0.88 |
| 1918 | 0.78 | 1 45 | 0.29 | 1 81 | 5.87 | 5.63 | 1.18 | 2.54 | 0.91 | 3.81 | 2.10 | 1.35 |
| 1919 | 0.08 | 3 00 | 3.67 | 5.30 | 2.96 | 7.36 | 2.68 | 2.19 | 7.47 | 2.20 | 3.84 | 0.93 |
| 1920 | 0.44 | 0.74 | 3.92 | 4.09 | 3.14 | 1.25 | 5.66 | 2.11 | 4.44 | 1.89 | 1.63 | 1.38 |
| 1921 | 0.59 | 0.92 | 1.07 | 3.72 | 3.62 | 4.66 | 2.49 | 6.63 | 7.16 | 1.51 | 0.35 | 0.80 |
| 1922 | 0.85 | 0.64 | 2.25 | 2.84 | 6.87 | 1.63 | 7.13 | 6.63 | 3.00 | 3.41 | 2.54 | 0.25 |
| 1923 | 0.88 | 0 36 | 4.34 | 1.76 | 4.78 | 4.95 | 0.78 | 5.34 | 5.17 | 1.10 | 0.55 | 0.61 |
| 1924 | 1.02 | 1.98 | 3.10 | 0.78 | 1.26 | 9.30 | 0.98 | 4.15 | 3.47 | 0.77 | 0.53 | 1.62 |
| 1925 | 0.23 | 0.50 | 0.88 | 1.64 | 0.77 | 6.40 | 2.21 | 4.79 | 3.75 | 3.22 | 0.32 | 1.67 |

of the interval. After a little preliminary reconnoitering an appropriate number of classes and their limits can be determined. Thus, in Table 3, the highest value noted was 10.91 and the lowest 0.08 (verify). The difference between these is 10.83, which suggests that if we took 20 classes we would have approximately a half inch as the width of a class interval. This, however, assumes we would start with 0.08 as our lower limit, which would give us awkward figures as limits. Therefore, our judgment suggests it would be better to start with 0 and continue by half-inch intervals as far as is necessary to take in the range of the given variates. We have estimated it will take approximatey 20 of these; actually it turns out to be 22. This number of intervals and their width is consistent with the general conditions (*a*) and (*b*) given above. On page 16 are given some supplementary rules which in general are helpful in making a frequency distribution.

**8. Distinction between Class Limits and Class Boundaries.** The pairs of numbers written in the column of classes of a frequency distribution are the lower and upper *class limits,* sometimes called open class limits. For instance, 1.00–1.49 are the limits of the third class of Table 4. When the measurements of Table 3 were made, readings were recorded to the nearest hundredth of an inch. Thus, a measurement which was more than 1.485 and less than 1.495 was recorded as 1.49. Likewise, if a measurement was more than 0.995 but less than 1.005, it would be recorded as 1.00. Therefore, the third class of Table 4 includes all measurements more than 0.995 and less than 1.495. These values are then the true or closed limits of the third class and are known as *class boundaries* or *end values.* A class boundary is the value halfway between the upper limit of one class and the lower limit of the next class. For example, the upper boundary of the fourth class of Table 4 is 1.995 which is the lower boundary of the fifth class. If we denote the variate values by $x$, the following table illustrates these remarks for the first five classes of Table 4.

| *Class Limits* | *End-x* | *Mid-x* |
|---|---|---|
| 0.00–0.49 | 0.495 | 0.245 |
| 0.50–0.99 | 0.995 | 0.745 |
| 1.00–1.49 | 1.495 | 1.245 |
| 1.50–1.99 | 1.995 | 1.745 |
| 2.00–2.49 | 2.495 | 2.245 |

The width of a class interval is the same, however, whether the

classes are expressed in terms of class limits or class boundaries, being the difference between the beginning of one class and the beginning of the next class. Similarly, the class mark as the mid-point of the interval is unaffected. Thus, for the class limits 1.00–1.49, the class mark is $\frac{1}{2}(1.00 + 1.49) = 1.245$; for the corresponding class boundaries, the class mark is $\frac{1}{2}(0.995 + 1.495) = 1.245$.

The distinction between class limits and class boundaries is an important one in plotting graphs, but in tabulating it is the class limits that should be expressed.

**9. Rules for Making a Frequency Distribution.**

(1) Determine the range of the table by finding the difference between the highest value and the lowest value among the items.
(2) Determine the number of equal parts into which the range shall be divided. The size of the class interval and the number of intervals depend upon the size and nature of the distribution. (Table 1 contains rather fewer classes than is usually desirable but an interval of 10 units is quite conventional in students' grades. An interval of 5 would be used if grades of A, A—, B, B—, etc., were given instead of A, B, etc.) Intervals of 0.5, 1, 2, 3, 5, 7, or 10 are the most common.
(3) Arrange a sheet with three headings: class interval, tally marks, frequency.
(4) Read off the items in the raw table and for each one record a mark, as shown in Table 2.
(5) Write the sum of the marks in each row in the frequency column. The sum of the frequencies should, of course, equal the total number of variates.

**10. Cumulative Frequencies.** The frequencies with which we have been concerned may be called absolute frequencies to distinguish them from two other kinds which will be mentioned in this course; namely, cumulative frequencies and relative frequencies. The first of these will be considered here.

Sometimes a statistical investigation is concerned with the number or percentage of variates which are "less than" or "more than" a given value. This is frequently the case in educational tests and in wage or salary statistics. Our chief interest in such cases may be the accumulated frequency of the several class intervals up to some class boundary. Hence we are led to form a cumulative frequency table. Such a table is built up by successively adding the several

(absolute) frequencies; thus: $f_1$, $f_1 + f_2$, $f_1 + f_2 + f_3$, etc., as illustrated in Table 7, where the data of Table 6 are used. We shall use $N$ to denote the sum of all the frequencies.

TABLE 6 — DISTRIBUTION OF INTELLIGENCE QUOTIENTS (IQ's) OF 905 SCHOOL CHILDREN FROM 5 TO 14 YEARS OF AGE. (DERIVED FROM L. M. TERMAN, *The Measurement of Intelligence*)

| *IQ* | *Number of Children* |
|---|---|
| 55– 64 | 3 |
| 65– 74 | 21 |
| 75– 84 | 78 |
| 85– 94 | 182 |
| 95–104 | 305 |
| 105–114 | 209 |
| 115–124 | 81 |
| 125–134 | 21 |
| 135–144 | 5 |

The cumulative frequency (*cum f*) at any class is the total (absolute) frequency up to the upper boundary of that class. This is the reason for placing the *cum f* entries opposite the *end-x* values and on lines between the *mid-x* entries. Thus, in the *cum f* column of Table 7, three students had IQ's less than 64.5, 24 less than 74.5, etc. The

TABLE 7 — CUMULATIVE DISTRIBUTION OF IQ'S (TABLE 6)

| *Class Mark Mid-x* | *Frequency f* | *Upper Boundary End-x* | *Cum f* | $\frac{\text{Cum } f}{N}$ |
|---|---|---|---|---|
| | | 54.5 | 0 | 0.000 |
| 59.5 | $3 = f_1$ | 64.5 | $3 = f_1$ | 0.003 |
| 69.5 | $21 = f_2$ | 74 5 | $24 = f_1 + f_2$ | 0.027 |
| 79.5 | 78 | 84.5 | 102 | 0.113 |
| 89.5 | 182 | 94.5 | 284 | 0.314 |
| 99.5 | 305 | 104.5 | 589 | 0.651 |
| 109.5 | 209 | 114.5 | 798 | 0.882 |
| 119.5 | 81 | 124.5 | 879 | 0.971 |
| 129.5 | 21 | 134.5 | 900 | 0.994 |
| 139.5 | 5 | 144.5 | $905 = N$ | 1.000 |

entries in the column headed $(cum\ f)/N$ give the percentages of the total frequency which are less than the values of the *end-x* column. Thus, from this column in Table 7, we can readily see that 88% of the children had IQ's less than 114.5 and only 11% less than 84.5.

Table 7 is known as a "less than" table. One could of course cumulate the frequencies from the bottom of the table, getting a "more than" distribution. The *cum f* column would then give the number of children whose IQ's are more than the values at the lower boundaries of the several class intervals.

The inverse operation to cumulating the frequencies is called "differencing" and is usually denoted by $\Delta$ (delta). If $S$ denotes any series of values, then $\Delta S$ denotes the results obtained by subtracting the first value of $S$ from the second value, the second from the third, etc. Differencing a column of cumulative frequencies obviously gives the absolute frequencies. Differencing a column of $(cum\ f)/N$ values gives the $f/N$ values.

## Exercises

1. What is the width of the class interval and the values of the class marks in Table 2?
2. Tabulate the grades of Table 1, using class intervals of 5 units.
3. With reference to Table 3, is it easy to answer such questions as the following:
   (*a*) In how many instances are the monthly rainfall between 2 inches and 3 inches?
   (*b*) In how many instances was the rainfall less than 5 inches?
   (*c*) What was the smallest monthly rainfall recorded?
   (*d*) What per cent of the total measured between 5 inches and 10 inches?
   (*e*) What measurement is the most common?
4. Refer to Table 4 and then answer the above questions.
5. Using your own judgment as to the most appropriate class interval, make a frequency distribution of the monthly rainfall for Des Moines from 1890 to 1925 (Table 5).
6. For Table 6 state the class boundaries (end values) and the class marks.
7. Difference the *cum f* column of Table 7.
8. Read the following references:
   (*a*) *Mathematics Essential for Elementary Statistics* — Walker, Chapter II.
   (*b*) *Standards and Requirements in Statistics* — Belcher. *Journal American Statistical Association*, vol. 21, p. 424.

**11. Additional Distributions.** The following distributions which will be referred to in subsequent chapters will serve as illustrative and laboratory material. They are not chosen on account of the importance of the data but merely to exemplify methods.

TABLE 8 — DISTRIBUTION OF LENGTHS OF 995 TELEPHONE CALLS. TIME IN SECONDS

| *Time* | *Number of Calls* |
|---|---|
| 0–99 | 1 |
| 100–199 | 28 |
| 200–299 | 88 |
| 300–399 | 180 |
| 400–499 | 247 |
| 500–599 | 260 |
| 600–699 | 133 |
| 700–799 | 42 |
| 800–899 | 11 |
| 900–999 | 5 |

(For future reference: $\bar{x} = 477.3$ secs., $\sigma = 148.5$ secs.)

TABLE 9 — DISTRIBUTION OF WEIGHT IN POUNDS AMONG 1000 8-YEAR-OLD GLASGOW SCHOOLGIRLS

| *Weight (mid-values)* | *Frequency* |
|---|---|
| 29 5 | 1 |
| 33 5 | 14 |
| 37.5 | 56 |
| 41 5 | 172 |
| 45 5 | 245 |
| 49 5 | 263 |
| 53 5 | 156 |
| 57 5 | 67 |
| 61 5 | 23 |
| 65 5 | 3 |

TABLE 10

Twelve dice were thrown 4096 times; only a throw of 6 was counted a success. The observed distribution follows:

| *Successes* $x_i$ | *Freque* $f_i$ |
|---|---|
| 0 | 447 |
| 1 | 1145 |
| 2 | 1181 |
| 3 | 796 |
| 4 | 380 |
| 5 | 115 |
| 6 | 24 |
| 7 | 7 |
| 8 | 1 |
| 9 | 0 |
| 10 | 0 |
| 11 | 0 |
| 12 | 0 |

(For future reference: $\bar{x} = 2$, $\sigma = 1.296$)

TABLE 11

Twelve dice were thrown 4096 times; a throw of 4, 5, or 6 points being reckoned a success. The following distribution was recorded:

| *Successes* | *Frequency* |
|---|---|
| 0 | 0 |
| 1 | 7 |
| 2 | 60 |
| 3 | 198 |
| 4 | 430 |
| 5 | 731 |
| 6 | 948 |
| 7 | 847 |
| 8 | 536 |
| 9 | 257 |
| 10 | 71 |
| 11 | 11 |
| 12 | 0 |

(For future reference: $\bar{x} = 6.139$, $\sigma = 1.712$)

TABLE 12 — FREQUENCY DISTRIBUTION OF THE WEIGHTS OF 1000 MALE STUDENTS (ORIGINAL MEASUREMENTS MADE TO NEAREST HALF POUND)

| *Class Pounds* | *Class Mark* | *Frequency* | *Cumulative Frequency* |
|---|---|---|---|
| 90– 99.5 | 94.75 | 2 | 2 |
| 100–109.5 | 104.75 | 21 | 23 |
| 110–119.5 | 114.75 | 104 | 127 |
| 120–129.5 | 124.75 | 196 | 323 |
| 130–139.5 | 134 75 | 248 | 571 |
| 140–149.5 | 144.75 | 197 | 768 |
| 150–159.5 | 154.75 | 133 | 901 |
| 160–169.5 | 164.75 | 47 | 948 |
| 170–179.5 | 174 75 | 25 | 973 |
| 180–189.5 | 184.75 | 14 | 987 |
| 190–199.5 | 194.75 | 7 | 994 |
| 200–209.5 | 204.75 | 4 | 998 |
| 210–219.5 | 214.75 | 0 | 998 |
| 220–229.5 | 224.75 | 0 | 998 |
| 230–239.5 | 234.75 | 1 | 999 |
| 240–249.5 | 244.75 | 1 | 1000 |

(For future reference: $\bar{x} = 138.65$, $\sigma = 18.03$, $\alpha_4 = .94$)

TABLE 13 — DISTRIBUTION OF SPAN (CENTRAL VALUES) IN INCHES AMONG 2000 ADULT MALES (ORIGINAL MEASUREMENTS TO THE NEAREST INCH)

| *Span* | *Frequency* | *Span* | *Frequency* |
|---|---|---|---|
| 58.5 | 1 | 71.5 | 217 |
| 59.5 | 2 | 72.5 | 176 |
| 60.5 | 1 | 73.5 | 132 |
| 61.5 | 6 | 74.5 | 82 |
| 62.5 | 7 | 75.5 | 48 |
| 63.5 | 22 | 76.5 | 20 |
| 64.5 | 55 | 77.5 | 16 |
| 65.5 | 111 | 78.5 | 12 |
| 66.5 | 146 | 79.5 | 3 |
| 67.5 | 182 | 80.5 | 1 |
| 68.5 | 229 | 81.5 | 2 |
| 69.5 | 265 | 82.5 | 1 |
| 70.5 | 263 | Total | 2000 |

The following references are recommended to those who desire some distributions which may be more interesting in themselves:

(*a*) Per cent Distribution of Deaths in Each Age Period, by Specified Causes. White Males and White Females, United States, 1942. Source: Metropolitan Life Insurance Company, *Statistical Bulletin*, October 1945, p. 7.

(*b*) Age of American Military Leaders. Source: Metropolitan Life Insurance Company, *Statistical Bulletin*, June, July, August, 1945.

(*c*) Employment Status of the Population by Age and Sex. Source: *Population, Third Series, The Labor Force*, Table 5, 16th Census.

(*d*) Distribution of Population by Age. Source: *Statistical Abstract*, 1943, p. 24.

# CHAPTER II

## GRAPHICAL REPRESENTATION

**1. The Function Concept.** Variables which are linked or related in some way are encountered in various fields of human experience. Several variables may be linked but we shall, for the present, consider the simple case where only two variables are involved. For example, the two related variables may be time and population, variate and frequency, rate of interest and accumulated principal, age and insurance premium. The primary purpose of a graph is to show diagrammatically how the values of one of two linked variables change with those of the other. One of the most useful applications of the graph occurs in connection with the representation of statistical data.

Underlying the intelligent use of graphs is the concept of *function*, which is a fundamental notion in mathematics and its applications. The mathematical meaning of function is a technical one, entirely different from the ordinary meaning. The student usually meets the word for the first time in algebra, when a linear or quadratic expression is spoken of as a function of $x$. An example is the equation

$$y = P(1 + x)^2.$$

The expression on the right is the function of $x$ ($P$ being constant) and for convenience it is denoted by the single letter $y$. Here $x$ is an interest rate and $y$ denotes the amount to which $P$ dollars will accumulate in two years at $x\%$ per year.

The statement that $y$ is a function of $x$ is written symbolically in the form

$$y = f(x).$$

This implies that a value of the function $y$ is determined when a value is assigned to the variable $x$. For this reason, $x$ is called the *independent variable* and $y$ the *dependent variable*. In place of $f$ other letters may be used. Thus, any one of the symbols

$$g(x), \quad h(x), \quad F(x), \quad \phi(x),$$

and so on, denotes a function of $x$. The same symbol may be used

to denote different functions in different problems, but different symbols are required to represent different functions in the same problem or discussion.

*Examples:*

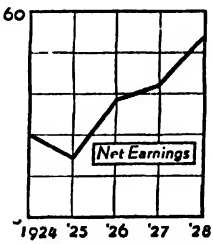
$$f(x) = 5x^2 - 3x + 2,$$
$$\phi(x) = Ke^{-x^2}.$$

Any mathematical expression involving a variable $x$ is a function of $x$. However, the word is often used to designate a relation that is completely divorced from any equation or expression. The central idea conveyed by this more general meaning is that of a correspondence between values of $y$ and values of $x$. The following definition is the result of a development over a long period and its formulation is due to Dirichlet, a famous French mathematician (1805–59).

DEFINITION. *Let there be a set of values assumed by the independent variable $x$. If to each $x$ in the set, there corresponds one or more values of $y$, then $y$ is said to be a function of $x$ in the set.*

It should be observed that this definition [1] is freed from any notion of the necessity of specifying the mathematical relation between $x$ and $y$. We may or may not know the special method by which the correspondence is set up. A mathematical formula or equation between $x$ and $y$ may not even exist. A function may thus be considered as being equivalent to a table in which one may look up any $x$ of the set of the definition, and find the corresponding $y$.

Much of the data in statistics comes under this general definition of function. Thus, in the following table, net earning is a function of the year, whether or not there is any equation defining that functional relationship.




| Year | Millions |
|---|---|
| 1924 | 45 |
| 1925 | 43 |
| 1926 | 49.6 |
| 1927 | 51.5 |
| 1928 | 57.3 |

Here the function is defined only for the indicated points which correspond to the values given in the table. The straight lines are drawn to help the reader visualize the relative positions of these values and not to represent the function at intermediate points. They may, however, be thought of as a first

[1] A classical example is the function which is defined for the infinite set of numbers from $x = 0$ to $x = 1$ to be unity for all rational numbers and zero for all irrational numbers.

approximation to the unknown function between the given values. Such a representation of the function could not, of course, be assumed in the case of discrete variates because then the function is discontinuous and does not exist except for the given values.

Referring again to the above definition, if there is only one value of $y$ corresponding to each value of $x$ then $y$ is called a *single-valued* function of $x$; otherwise $y$ is said to be a *multiple-valued* function of $x$. Child weight would be an example of a multiple-valued function of age, being different for different children. The weight of a particular child would be a single-valued function of age. For the most part we shall be concerned with single-valued functions.

**2. Charts.** A detailed study of the technique of representing data by broken lines, by charts or bar graphs, etc., will not be undertaken here. It is a rather specialized and non-mathematical subject, and the student interested in plain-scale cartography can readily find books on the subject which are very readable.[1] (A discussion of ratio charts is given in Chapter VII.)

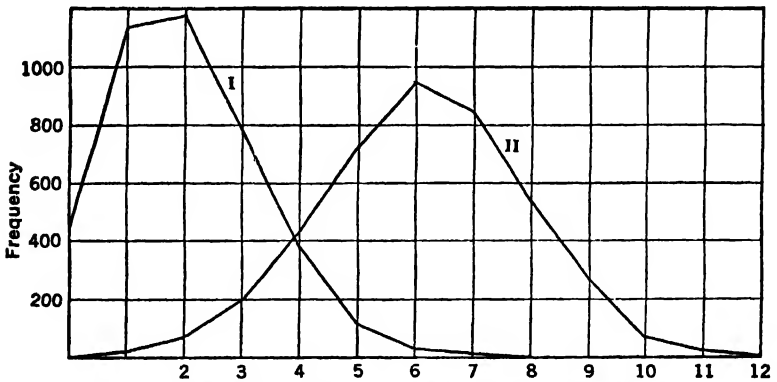



FIG. 1 — FREQUENCY POLYGONS FOR DISTRIBUTIONS OF DISCRETE VARIATES

**3. Frequency Polygon.** We present now a discussion of the graphs that are used in connection with frequency distributions. A

[1] For example,

(*a*) *Graphs: How to Make and Use Them* — H. Arkin and R. Colton. 2nd ed. Harper.

(*b*) *Engineering and Scientific Graphs for Publication.* American Standards Association, New York.

(*c*) Reference 8 in our Introduction.

distribution of discrete variates may be represented graphically by plotting the points $(x_1, f_1)$, $(x_2, f_2)$, $\cdots$ $(x_k, f_k)$, and drawing a broken line through them. Such a graph is called a frequency polygon because it is a polygon formed by connecting the tops of a series of ordinates whose lengths are proportional to the various frequencies and whose abscissas correspond to the variate values of the distribution. Figure 1 will serve as an illustration. For a table of discrete variates the function exists only for the given values. Likewise, its graph is discontinuous. The straight lines connecting the points serve merely to "carry the eye," thus giving a better idea of the shape and position of the distribution.

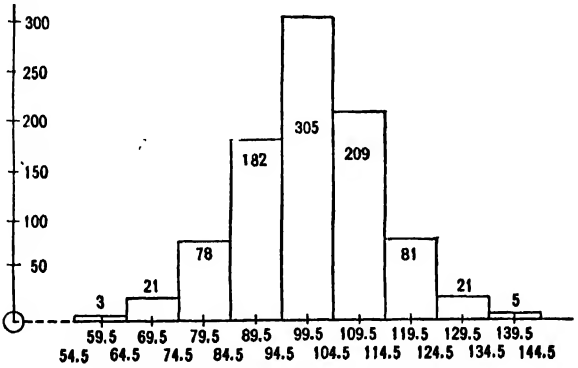



FIG. 2 — HISTOGRAM FOR TABLE 6

**4. Histogram.** If the frequency distribution is one of grouped variates (discrete or continuous) it is better to use some form of graphical representation which recognizes the fact that the several measurements in a table do not lie precisely at the class marks but are spread out over the intervals of which the class marks are centers. This may be accomplished through the use of a *histogram*. A histogram is a series of rectangles erected at the class boundaries with altitudes proportional to the respective class frequencies, and centered on the class marks. Thus the frequencies are represented by areas. (See Figure 2.) If the bases are all of unit length then the altitudes are also equal to the frequencies. The histogram is an important and useful graphical device for representing frequency distributions.

**5. Frequency Curves.** The shape of the distribution may be emphasized by constructing a continuous frequency curve such that

the areas under the curve beween the ordinates at the upper and lower boundaries of the various rectangles will equal approximately the areas of the corresponding rectangles. Thus, in Figure 3, the area of all the rectangles represents the total frequency 1000, and the area of the three rectangles labeled A, B, C represents the number of individuals weighing between 139.75 pounds and 169.75 pounds. The dotted line represents roughly the frequency curve corresponding to the histogram.

Representing each class frequency of a distribution of continuous variates by a rectangle is equivalent to saying that we realize that

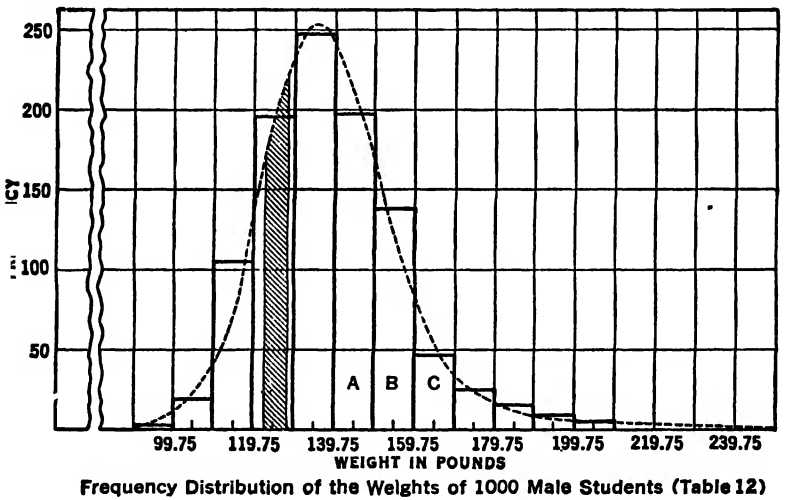



FIG. 3 — HISTOGRAM AND FREQUENCY CURVE FOR A DISTRIBUTION OF CONTINUOUS VARIATES

the function exists for points other than the class marks, but we do not know what it is for these points, and so as a first approximation we assume that the variates are uniformly distributed over each interval, which is equivalent to regarding them as concentrated at the class marks. If the class intervals were made smaller and smaller and at the same time the number of variates were proportionally increased, the upper bases of the rectangles would approach more and more the frequency curve which represents the ideal or theoretical mathematical function relating frequency with variate value for the given distribution.

A frequency curve is often drawn for convenience in describing

the properties of an observed distribution, although strictly speaking, the concept of a frequency curve is applicable only to an infinite "universe" of continuous variates. The data at hand are supposed to be a "sample" from the universe represented by the frequency curve.

The more common types of distributions may be represented by bell-shaped curves which are either symmetrical or skew. For elementary purposes it is sufficient to consider frequency distributions as of these two general types. In passing, we may also mention two other types which are known as J-shaped and U-shaped. For examples of these types see Yule and Kendall, *An Introduction to the Theory of Statistics*, Ch. VI.

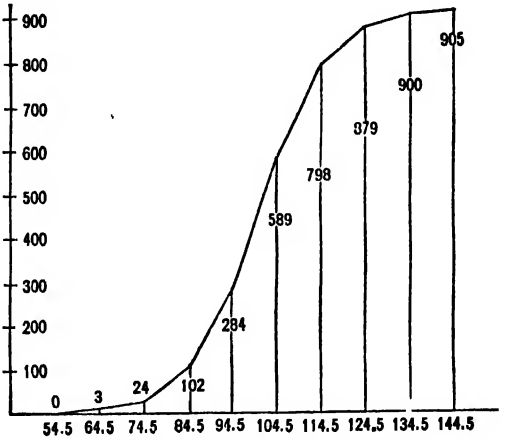



FIG. 4 — OGIVE FOR TABLE 7

**6. Ogives.** The graphs of cumulative frequencies are called *ogives*. The ogive for Table 7 is shown in Figure 4 and is constructed by plotting the points (54.5, 0), (64.5, 3), etc., as in algebra, and joining them with straight lines.

The student should observe that while *cum f* is a function of $x$ it is defined for the *end-x* values only. Occasions will perhaps arise when we desire the $x$-value corresponding to some intermediate *cum f* value, say 453 in Figure 4 Conversely, we might wish to know the *cum f* value for some intermediate $x$-value, say at $x = 97$. Strictly speaking, we do not know the answer in either case, inasmuch as we do not know how the IQ's are distributed over the interval. Perhaps all the individual values in the interval 94.5–104.5 (say) are

less than 97; perhaps none are. The fairest assumption we can make is that they are uniformly distributed throughout the interval. This means graphically that we represent *cum f* over each interval by a straight line, as is done in Figure 4. We may now interpolate under this line for intermediate values. This is "straight line interpolation" and is what the student uses when he interpolates in logarithms.

More refined methods exist for interpolating values of a function between the observed values but their study constitutes a separate branch of mathematics beyond the scope of this course. It should be observed that the straight line used here is a first approximation to the unknown function, and not merely a device to carry the eye as in the case of a frequency polygon for a discontinuous distribution of discrete variates.

**7. Relation of *Cum f* to Areas.** The sum of the frequencies (*cum f*) up to any value of $x$ means, graphically, the sum of the areas of the rectangles of the histogram up to that value. Thus in Figure 4, the ordinate erected at $x = 84.5$ represents the sum of the frequencies $(3 + 21 + 78) = 102$ (Figure 2). If a frequency curve represents the distribution, then *cum f*, corresponding to any value of $x$, is the area under the curve up to that value. Thus, in Figure 3, *cum f* corresponding to $x = 139.75$ is approximately the area under the smooth curve up to $x = 139.75$, and the total area under the curve is *cum* $f = N$.

## Exercises

1. (*a*) If $f(x) = 2x^3$ exhibit $f(-x)$. Give the value of $f(3)$, of $f(-2)$.
   (*b*) Let $f(x)$ denote a given function which is defined for all real values of $x$ under consideration so that if $c$ is any admissible number $f(c)$ is defined. What is the graphical meaning of $f(c)$?
2. If $\phi(x) = Ke^{-x^2}$, (*a*) show that $\phi(x) = \phi(-x)$; (*b*) give the value of $\phi(0)$.
3. If $h(x) = ax^2 + bx + c$, and $h(x) = h(-x)$, show that $b = 0$.
4. If $f(x) = u^x$, show that $f(u) \times f(v) = f(u + v)$.
5. If $g(x) = \log\{(1 - x)/(1 + x)\}$, show that $g(u) + g(v) = g\{(u + v)/(1 + uv)\}$.
6. Make a histogram for the data of Table 4.
7. Same as exercise 6 for Table 8 or 9.
8. Construct an ogive for the cumulative frequencies given in Table 12.
9. Find the cumulative frequencies and construct the ogive for Table 9.
10. For further discussion of ogive curves and their uses, read the following references:
    (*a*) *Elements of Statistics* — Davis and Nelson, pp. 23–28.
    (*b*) *The Mathematics of Statistics* — Burgess, pp. 61–72.

# CHAPTER III

## AVERAGES

**1.** It was pointed out in Chapter I that classification of the variates of any long series is the first step necessary to overcome the confusion of detail in the original observations, and to make comparisons with other distributions possible. In Chapter II graphical methods were studied which describe, to some extent, the shape and position of the distribution. Although these methods are helpful, their contribution is largely qualitative.

It is desirable to formulate quantitative descriptions for characterizing a distribution, and as an aid in this direction *averages* are very useful. They are also called *measures of location*. An average is a quantity locating a central value of the distribution. In a sense, it is a typical value of the whole set of variates, although it is not necessary that it actually have the value of one of the items of the set it represents. There are five averages in common use. These are: *arithmetic mean*, *mode*, *median*, *geometric mean*, and *harmonic mean*. The means and median are most frequently used although the arithmetic mean is by far the most important in general statistical work, and the others are of service in special cases. We will consider them in the order named. First, however, it will be desirable to discuss certain symbols and notation which will facilitate the development of formulas.

**2. Notation.** If $x$ denotes a variable, then $x_1, x_2, \cdots, x_N$, are general symbols for the values which $x$ may take. When we are concerned with a sum like the following,

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_i + \cdots + x_N,$$

it is customary to designate it by placing the Greek capital letter $\sum$ (sigma) before the general term, thus

$$\sum_{i=1}^{N} x_i = x_1 + x_2 + \cdots + x_i + \cdots + x_N.$$

The symbol $\sum$ is a sort of mathematical verb and the notation written above and below it may be called adverbs. Mathematicians

call $\sum$ an *operator* and speak of the "adverbs" as *limits*. When $\sum_{i=1}^{N}$ is placed before any quantity, it means, "add up all quantities like $\cdots$ which are formed by giving $i$ the values of every positive integer from $i = 1$ to $i = N$, inclusive." Thus if $x_i$ stands for "variates" in Table 1, $x_1$ refers to the first value 75, $x_2$ refers to the second value 80, etc., and $x_N$ refers to the last value 56. Here $N = 100$. Hence the compact notation $\sum_{i=1}^{100} x_i$ denotes the sum of all the variates in Table 1. The symbol $\sum_{i=1}^{N} x_i$ is read, "the summation of $x$-sub-$i$, $i$ varying (or running) from one to $N$." The subscript $i$ is called the *index* of summation. Any letter may be used as an index but it is conventional to use $i$ or $j$. Also the upper limit may be denoted by any letter but we shall use $N$ to denote the total number of variates (some of which may be alike) in a set.

If a variable $x$ is to take on the particular values, 1, 2, 3, etc., instead of the general values $x_1$, $x_2$, $x_3$, etc., then $x$ itself becomes the index of summation and we write $x = 1$ underneath $\sum$. Thus

$$\sum_{x=1}^{N} x = 1 + 2 + 3 + \cdots + N,$$

$$\sum_{x=1}^{N} x^2 = 1 + 2^2 + 3^2 + \cdots + N^2.$$

Frequently the index of summation is understood from the context and the notation at the top and bottom of $\sum$ may be omitted if no ambiguity results.

It is imperative that the student master, as soon as possible, the significance and utility of the $\sum$ notation.

*Illustrations:*

1. $\sum_{i=1}^{N} 3x_i = 3x_1 + 3x_2 + \cdots + 3x_N$
$= 3(x_1 + x_2 + \cdots + x_N).$

2. $\sum_{i=1}^{5} (x_i + c) = (x_1 + c) + (x_2 + c) + (x_3 + c)$
$+ (x_4 + c) + (x_5 + c)$
$= (x_1 + x_2 + x_3 + x_4 + x_5) + 5c.$

3. $\sum_{i=1}^{4} x_i f_i = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4.$

4. $\sum_{j=1}^{4} x_j y_j = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$

5. $\sum_{u=1}^{N} u^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2.$

The following simple theorems will be useful in our work.

**Theorem I.** *The summation $\sum$ of an algebraic sum of two or more terms is the same algebraic sum of the $\sum$'s of these terms taken separately. In symbols:*

$$\sum_{i=1}^{N}(x_i + y_i - z_i) = \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} z_i.$$

**Theorem II.** *A constant factor may be removed from under the summation sign and written outside as a factor. Thus,*

$$\sum_{i=1}^{N} cx_i = c\sum_{i=1}^{N} x_i.$$

*Proofs:* It is left as an exercise for the student to prove these two theorems by expanding the expressions.

**Theorem III.** *If the expression under $\sum_{i=1}^{N}$ is a constant c, the expanded result is Nc.*

*Examples:*

1. $\sum_{i=1}^{N} c = c + c + \cdots + c = Nc.$

2. $\sum_{i=1}^{N}(x_i - c) = \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} c$, by Theorem I

$$= \sum_{i=1}^{N} x_i - Nc, \text{ by Theorem III.}$$

The above theorems hold also if we replace the notation

$$\sum_{i=1}^{N} x_i \text{ by } \sum_{x=1}^{N} x, \text{ etc.}$$

The next two theorems have to do with summing integers. The numbers used in counting,

$$1, 2, 3, 4, 5, \cdots$$

are called integers or natural numbers.

**Theorem IV.** *The sum of the first N integers is*

$$\frac{N(N+1)}{2}.$$

*In symbols:* $$\sum_{x=1}^{N} x = \frac{N(N+1)}{2}.$$

This result follows from the fact that the integers form an arithmetic progression.

**Theorem V.** *The sum of the squares of the first N integers is*

$$\frac{N(N+1)(2N+1)}{6}.$$

*In symbols:* $$\sum_{x=1}^{N} x^2 = \frac{N(N+1)(2N+1)}{6}.$$

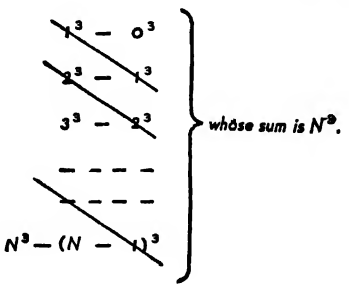
*Proof:* Let us take the identity $x^3 - (x-1)^3 = 3x^2 - 3x + 1$, and sum each side for $x = 1$ to $N$. Thus,

$$\sum_{x=1}^{N}[x^3 - (x-1)^3] = \sum_{x=1}^{N}[3x^2 - 3x + 1].$$

Applying Theorems I–III to the right member we have

$$\sum_{x=1}^{N}[x^3 - (x-1)^3] = 3\sum_{x=1}^{N} x^2 - 3\sum_{x=1}^{N} x + N.$$

Performing the indicated sum in the left member, we have




Therefore $$N^3 = 3\sum x^2 - 3\sum x + N.$$

Hence, using Theorem IV and simplifying,

$$\sum_{x=1}^{N} x^2 = \frac{2N^3 + 3N(N+1) - 2N}{6},$$

whence $$\sum_{x=1}^{N} x^2 = \frac{N(N+1)(2N+1)}{6}$$

**3. Arithmetic Mean.** The arithmetic mean of a set of variates is defined as the sum of the variates divided by their number. We are thinking now of a set of ungrouped variates, like that of Table 1. If we use the symbol $\bar{x}$ to represent the arithmetic mean of the $N$ variates $x_1, x_2, x_3 \cdots, x_N$, then

$$\bar{x} = \frac{1}{N}(x_1 + x_2 + x_3 + \cdots + x_N),$$

or using the more compact notation of the preceding section, we have

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{1}$$

Each item in the set is thus represented in the arithmetic mean in proportion to its magnitude.

As an illustration, it is easily verified that for the set of grades given in Table 1,

$$\bar{x} = \frac{7267}{100} = 72.67.$$

Computing the mean [1] strictly according to definition (1) may be called the serial method to distinguish it from other methods which will be presented. This definition is applicable when $N$ is so small that a grouping of the variates into a frequency distribution is not feasible.

If $x$ refers to the integers from 1 to $N$ their mean is

$$\bar{x} = \frac{1}{N}\sum_{x=1}^{N} x. \tag{1a}$$

**4. Weighted Arithmetic Mean.** It will be noticed that several of the grades given in Table 1 are alike. For example, 80 occurs seven times. It should be evident that the same result would be found for the mean if, instead of summing the individual values, each value was first multiplied by the frequency with which it occurs and all such products were then added. In general, if the values $x_1, x_2, \cdots, x_k$ occur with corresponding frequencies $f_1, f_2, \cdots, f_k$, respectively,

[1] When there is no ambiguity, the arithmetic mean is often referred to as the mean.

where $f_1 + f_2 + \cdots + f_k = N$, it follows that

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_k f_k}{f_1 + f_2 + \cdots + f_k},$$

or, in shorter notation

$$\bar{x} = \frac{1}{N}\sum_1^k f_i x_i, \text{ where } N = \sum_1^k f_i. \tag{2}$$

When obtained in this way, $\bar{x}$ is generally called a weighted arithmetic mean. The term originated in experimental science where some readings which have been made under more favorable conditions are " weighted " according to their reliability or importance. When the weights have been chosen, they become, essentially, frequencies.

If the $x$'s are added individually, the $f$'s become unity, and equation (2) reduces to (1). The student should notice that, for the same data, $\sum_1^k f_i x_i$ is numerically equal to $\sum_1^N x_i$. He should also observe that $N$ refers to the number of variates in the set (some of which may be alike), whereas $k$ refers to the number of *different* values of $x$ in the set and hence to the number of products of the form $x_i f_i$ where $f_i$ is the number of times $x_i$ occurs. In the following example, $N = 8$ and $k = 4$.

*Example.* For the values 6, 8, 7, 6, 5, 7, 6, 5,

$$\sum_{i=1}^8 x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 6 + 8 + 7 + 6 + 5 + 7 + 6 + 5 = 50.$$

$$\sum_{i=1}^4 f_i x_i = f_1 x_1 + f_2 x_2 + f_3 x_3 + f_4 x_4 = 2 \cdot 5 + 3 \cdot 6 + 2 \cdot 7 + 1 \cdot 8 = 50. \quad \sum_{i=1}^4 f_i = 8.$$

By either method, $\bar{x} = 50/8 = 6.25$.

## Exercises

1. Write in expanded form:

   (a) $\sum_{i=1}^k x_i f_i$; (b) $\sum_{i=1}^k x_i^2 f_i$; (c) $\sum_{i=1}^k (x_i - \bar{x}) f_i$.

2. Write in expanded form:

   (a) $\sum_1^{n_1} f_i$; (b) $\sum_{i=n_1+1}^{n_1+n_2} f_i$; (c) $\sum_{i=1}^{n_1} x_i f_i + \sum_{i=n_1+1}^{n_1+n_2} x_i f_i$

3. Express 2(c) as a single summation, if $n_1 + n_2 = k$.

4. Write in the abbreviated form, using $\sum$:

   (a) $x_1f_1 + x_2f_2 + \cdots + x_kf_k$.

   (b) $(x_1 - \bar{x})f_1 + (x_2 - \bar{x})f_2 + \cdots + (x_k - \bar{x})f_k$.

   (c) $\frac{1}{N}[(x_1 - \bar{x})^2f_1 + (x_2 - \bar{x})^2f_2 + \cdots + (x_k - \bar{x})^2f_k]$.

5. Prove:

   (a) $\sum_1^k (x_i + 1)^2f_i = \sum_1^k x_i^2f_i + 2\sum_1^k x_if_i + N$.

   (b) $\sum_{x=0}^{n} x(x-1)p = \sum_{x=2}^{n} x(x-1)p$.

6. Compute the value of exercise 1(c) for the example in §4, using the following form:

| $x_i$ | $f_i$ | $(x_i - \bar{x})$ | $(x_i - \bar{x})f_i$ |
|---|---|---|---|
| 5 | 2 | −1.25 | −2.50 |
| 6 | 3 | | |
| 7 | 2 | ? | ? |
| 8 | 1 | | |
| | | | $\sum(x_i - \bar{x})f_i = ?$ |

7. Distinguish between $\sum_{i=1}^{N} x_iy_i$ and $\left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)$. Write in expanded form.

8. (a) Express in $\sum$ notation: Each different variate is multiplied by its own $f$ and the sum of the results is divided by $N$.

   (b) Give word statements of the expressions in Exercise 4.

   (c) Express the general polynomial of degree $n$ in $x$,

$$a_0 + a_1x + a_2x^2 + \cdots + a_n$$

   in $\sum$ notation.

9. Using the identity

$$x^2 - (x-1)^2 = 2x - 1$$

derive the result

$$\sum_{x=1}^{N} x = \frac{N(N+1)}{2}$$

by a method analogous to the proof of Theorem V.

10. (a) Express in abbreviated notation: The sum of the squares of the $x$'s divided by the square of their sum.

    (b) If $x$ refers to the integers from 1 to $N$, evaluate your answer to (a) in terms of $N$.

    (c) Show that the mean of the first $N$ integers is $(N + 1)/2$.

**5. Arithmetic Mean from Frequency Table.** The variates in each class interval of a frequency distribution are assumed to have the value of the class mark for that interval. Therefore, we may use

formula (2) to find the mean of a frequency distribution. In this case, $x_i$ represents the mid-value of the $i$th class interval, $f_i$ the corresponding frequency, and $k$ the number of intervals; $i$ running from 1 to $k$. The method of applying (2) is illustrated in Table 14 from the data of Table 2.

TABLE 14

| *Class Interval* | *Class Mark* $x$ | *Frequency* $f$ | *Product* $fx$ |
|---|---|---|---|
| 30–39 | 34.5 | 2 | 69.0 |
| 40–49 | 44.5 | 3 | 133.5 |
| 50–59 | 54.5 | 11 | 599.5 |
| 60–69 | 64.5 | 20 | 1290.0 |
| 70–79 | 74.5 | 32 | 2384.0 |
| 80–89 | 84.5 | 25 | 2112.5 |
| 90–99 | 94.5 | 7 | 661.5 |
| Totals | | $\sum f = 100$ | $\sum fx = 7250.0$ |

$$\bar{x} = \frac{7250}{100} = 72.50.$$

If we denote the class interval by $c$ then it is obvious that $c = 10$ in Table 14.

In this connection it is interesting to note that our result here differs very little from the true value 72.67 and therefore our assumption that all values in a given class may be taken as the class mark seems to cause little error in the result obtained for the mean. This can be proved mathematically (under certain assumptions) and will be referred to later.

**6. Translation of Axes; Deviations.** It is frequently useful to employ the methods and results of geometry in connection with the problems of statistics. Foremost among these methods is the representation of numbers by points on a line; an origin and a unit of measure having been chosen, a coördinate is assigned to each point on the line. When a frequency distribution is represented by a graph, we have seen in Chapter II that the variate values are used as abscissas or measurements along the $x$-axis. The mean is therefore the point on the $x$-axis whose coördinates are $(\bar{x}, 0)$. Its position may be emphasized by drawing a vertical line through this point, but it is

the horizontal distance of the point from the origin and not the vertical line which represents graphically the mean.

In discussing the variates we may often work with smaller numbers by changing the origin of reference. If new axes, $x'y'$, are taken parallel to the old axes, $xy$, with positive directions preserved, the axes are said to be *translated* from one position to the other. A translation of axes corresponds to a transformation of coördinates. Thus if we let

$$x' = x - x_0, \quad y' = y - y_0$$

the origin is translated to the point $(x_0, y_0)$. Since the variates are denoted by $x$ we are concerned here only with the transformation $x' = x - x_0$ which translates the origin to the point $(x_0, 0)$. The variates referred to a new origin are often called *deviations*. In particular if we translate the origin to the mean by letting

$$x' = x - \bar{x},$$

then for a frequency distribution the deviations are the values obtained by subtracting $\bar{x}$ from each of the class marks. Thus,

$$\begin{aligned} x_1' &= x_1 - \bar{x} \\ x_2' &= x_2 - \bar{x} \\ &\cdots \\ x_k' &= x_k - \bar{x}. \end{aligned}$$

The units of measurement remain unchanged. Figure 5 shows the two systems when the axes are translated to $(\bar{x}, 0)$. Obviously, any

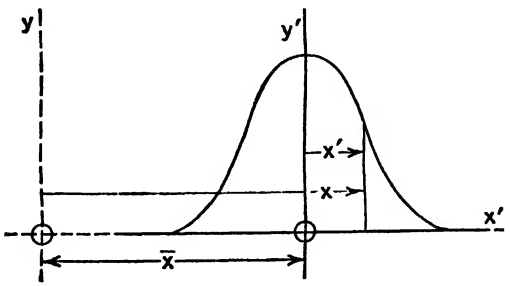



FIG. 5

variates that are larger than $\bar{x}$ will be positive in terms of $x'$ and any variates smaller than $\bar{x}$ will be negative in terms of $x'$.

**7. Properties of $\bar{x}$.** There are two important properties of $\bar{x}$ which may be stated in the following theorems:

**Theorem VI.** *The algebraic*[1] *sum of the deviations of all the variates from their arithmetic mean is zero.*

*Proof:* Let $x'$ represent a deviation from the mean. Multiplying each different deviation by the number of times it occurs and adding these products we have,

$$\sum_1^k f_i x_i' = \sum_1^k f_i(x_i - \bar{x})$$
$$= \sum_1^k f_i x_i - \sum_1^k f_i \bar{x}, \text{ by Theorem I}$$
$$= \sum_1^k f_i x_i - \bar{x}\sum_1^k f_i, \text{ by Theorem II.}$$

Recalling from (2) that $\sum_1^k f_i x_i = N\bar{x}$, and that $\sum_1^k f_i = N$, we have

$$\sum_1^k f_i(x_i - \bar{x}) = N\bar{x} - \bar{x}N = 0. \tag{3}$$

**Theorem VII.** *If the variates are referred to a new origin $x_0$ and expressed in units of $c$ by means of the transformation*

$$u = \frac{x - x_0}{c}, \ (c \neq 0), \tag{4}$$

*then the old mean, $\bar{x}$, is related to the new mean, $\bar{u}$, by the following formula:*

$$\bar{x} = c\bar{u} + x_0. \tag{5}$$

*Proof:* From (4),

$$x = cu + x_0 \tag{4a}$$

and substitution of this value for $x$ in definition (2) gives

$$\bar{x} = \frac{1}{N}\sum_1^k f_i(cu_i + x_0).$$

By Theorems I and II this equals

$$\frac{c}{N}\sum_1^k f_i u_i + \frac{x_0}{N}\sum_1^k f_i.$$

[1] That is, taking account of signs. Some of the deviations will be positive and some negative.

But the first of these expressions is, by definition, $c$ times the mean value of $u$, and the second is, from (2), simply $x_0$. Therefore

$$\bar{x} = c\bar{u} + x_0.$$

This is an important relation and its derivation should be mastered. Observe that the size of a $u$-unit will be $c$ times as large as the size of an $x$-unit.

COROLLARY. *If the mean of the deviations of the variates from any arbitrary number, $x_0$, is found and added algebraically to $x_0$, the result is the mean $\bar{x}$. In symbols,*

$$\bar{x} = \frac{1}{N}\sum_{1}^{k} f_i(x_i - x_0) + x_0. \tag{6}$$

The proof follows from (4) and (5).

In (5) and (6), $x_0$ may be regarded as a provisional mean, and the first term in the right members may be thought of as the correction to be added algebraically to the provisional mean in order to get the true mean.

**8. Short Methods of Computing $\bar{x}$.** In certain cases, the method of computing the mean by (2), as shown in Table 14, can be simplified by use of Theorem VII.

*Case I (class intervals equal).* If the class marks are equispaced, let $c$ equal the class interval and choose $x_0$ as one of the class marks, usually the one opposite the largest frequency. From (4), $x_0$ becomes the origin of $u$, because when $x = x_0$, $u = 0$.

The method of using (5) is illustrated in Table 15, page 40. Here $c = 10$ and we choose $x_0 = 74.5$, so (4) becomes

$$u = \frac{x - 74.5}{10}.$$

Substituting the given values of $x$ in this relation we get the values in the $u$ column. So in running the $fu$ column, small values of $u$ are multipliers of the larger values of $f$. Then

$$\bar{u} = \frac{1}{100}\sum fu = \frac{-20}{100} = -.2,$$

so from (5),

$$\bar{x} = 10(-.2) + 74.5 = 72.5\%.$$

It should be evident that the final value obtained for the mean is independent of the choice of the arbitrary value $x_0$. This choice is only a rough guess and it is really immaterial which of the given values is selected as $x_0$, except that the nearer it is to the mean the lighter will be the calculations to follow. A *check* on the arithmetic may, therefore, be effected by selecting a different provisional mean.

TABLE 15 — MEAN OF 100 GRADES USING CLASS INTERVAL AS UNIT

| $x$ | $u$ | $f$ | $fu$ |
|---|---|---|---|
| 34.5 | −4 | 2 | − 8 |
| 44.5 | −3 | 3 | − 9 |
| 54.5 | −2 | 11 | −22 |
| 64.5 | −1 | 20 | −20 |
| 74.5 | 0 | 32 | 0 |
| 84.5 | 1 | 25 | 25 |
| 94.5 | 2 | 7 | 14 |
| Totals | | 100 | −20 |

This indirect method is sometimes called *coding* because the variates are coded to another scale in which it is easier to compute the mean. Formula (5) is the relation, then, for transforming the mean from one scale to another.

If one's statistical interests are limited to computing means, then (2) cannot be improved upon if calculating machines are to be used. It should be understood, however, that techniques must be developed now for subsequent purposes. The indirect method is part of a pattern which is useful in later chapters. From this standpoint, one should practice using it at this stage when $N = \sum_1^k f_i$ is large and the $x$'s are equispaced.

*Case II* (*class intervals unequal*). Occasionally a frequency distribution is encountered in which the variates are not equispaced; it is then usually best to take $c = 1$ (unless the $x$'s have a common factor $c$) and be content with whatever simplification results from a suitable choice of $x_0$. This is equivalent to using the above corollary.

In Table 16, we choose $x_0 = 200$ and are thus able to simplify the work a little. (See page 41.)

TABLE 16

| $x$ | $f$ | $u$ | $uf$ |
|---|---|---|---|
| 106.12 | 7 | − 93.88 | − 657.16 |
| 191.83 | 14 | − 8.17 | − 114.38 |
| 246.48 | 32 | 46.48 | 1487.36 |
| 283.63 | 49 | 83.63 | 4097.87 |
| 257.65 | 55 | 57.65 | 3170.75 |
| 294.51 | 54 | 94.51 | 5103.54 |
| 222.53 | 35 | 22.53 | 788.55 |
| 71.43 | 14 | −128.57 | −1799.98 |
| Totals | 260 | | 12076.55 |

$$\bar{u} = \frac{1}{N}\sum f_i u_i = \frac{1}{N}\sum f_i(x_i - 200)$$

$$= \frac{12076.55}{260} = 46.448$$

$$\bar{x} = \bar{u} + x_0 = 246.45.$$

---

**9. Geometric Explanation.** Let us consider further the relation between the variables $x$ and $u$, defined by the expression.

$$u = x - x_0 \tag{4}$$

A geometric explanation will be helpful.

Graphically, the $x$ values are distances along the $x$-axis measured from zero as origin. Likewise $x_0$ is some point on the $x$-axis at a distance of $x_0$ units from zero. If now the points representing the $x$ values are measured from $x_0$ as origin they are denoted by $x - x_0$. (See Figure 6.) Thus if $x_0 = 24$, a value which is 36 with reference to the origin of $x$ will be 12 with reference to $x_0$; likewise a value $x = 18$ becomes $x - x_0 = -6$ when referred to $x_0$ as origin. It

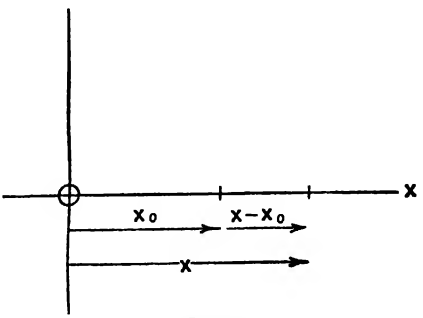



FIG. 6

should be noted that $x - x_0$ is in the same units as $x$. Thus if $x$ is in inches, $x - x_0$ will also be in inches. But $(x - x_0)/12$ would then be in feet. Instead of dividing by 12 suppose we divide by $c$. Then $(x - x_0)/c$ will be in units of $c$ whatever $c$ may be. It is convenient to denote the resulting values by a different letter, say $u$. Therefore the numerator of (4) changes the origin of reference but does not affect the scale of measurement. The denominator changes the scale, there being $c$ of the $x$ units in one of the $u$ units. Relation (4) has this generalized meaning apart from statistics. Mathematical notation is applicable to many different fields of knowledge. A relation like (4) which occurs in physics is $C = (5/9)(F - 32)$; it connects temperature on the Centigrade and Fahrenheit scales.

When (4) is applied to a frequency distribution it is convenient to select $x_0$ as one of the *mid-x* values and to take $c$ as the width of the class intervals. Under Case I, the mean is found with reference to $x_0$ and in units of $c$. This is the mean, $\bar{u}$, of the numbers representing the various class intervals weighted with the corresponding frequencies. After this mean is computed it may be converted back into units of $x$ by multiplying by $c$, and then referred to the origin of $x$ by adding $x_0$. (See Figure 7.) Hence we have $\bar{x} = c\bar{u} + x_0$. Thus we arrive at the same result as that obtained algebraically.

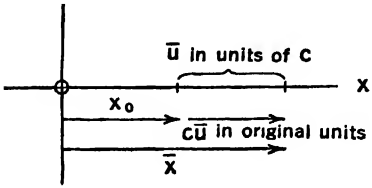



FIG. 7 — If $x_0 < \bar{x}$, $cu$ is positive; if $x_0 > \bar{x}$, $cu$ is negative.

If we had denoted the variates by $y$ we could have used the relation

$$v = \frac{y - y_0}{c}$$

corresponding to (4). Geometrically, this would mean a change of units and a translation of origin in the $y$-direction. The relation corresponding to (5) would then be

$$\bar{y} = c\bar{v} + y_0$$

where $\bar{v} = \dfrac{1}{N}\sum f_i v_i$.

As the short-cut method is an important one, another illustration is given in Table 17 (based on Table 4). Here we take $u = (x - 2.745)/0.5 = 2(x - 2.745)$.

TABLE 17 — COMPUTATION OF MEAN MONTHLY RAINFALL AT IOWA CITY 1890–1925

| $x$ | $f$ | $u$ | $fu$ |
|---|---|---|---|
| 0.245 | 23 | −5 | −115 |
| 0.745 | 42 | −4 | −168 |
| 1.245 | 58 | −3 | −174 |
| 1.745 | 62 | −2 | −124 |
| 2.245 | 49 | −1 | − 49 |
| 2.745 ← $x_0$ | 47 | 0 | 0 |
| 3.245 | 32 | 1 | 32 |
| 3.745 | 27 | 2 | 54 |
| 4.245 | 18 | 3 | 54 |
| 4.745 | 15 | 4 | 60 |
| 5.245 | 14 | 5 | 70 |
| 5.745 | 7 | 6 | 42 |
| 6.245 | 10 | 7 | 70 |
| 6.745 | 5 | 8 | 40 |
| 7.245 | 6 | 9 | 54 |
| 7.745 | 5 | 10 | 50 |
| 8.245 | 3 | 11 | 33 |
| 8.745 | 2 | 12 | 24 |
| 9.245 | 5 | 13 | 65 |
| 9.745 | 0 | 14 | 0 |
| 10.245 | 1 | 15 | 15 |
| 10.745 | 1 | 16 | 16 |
| Totals | 432 | | 49 |

$$\bar{x} = 2.745 + \frac{(0.5)(49)}{432}$$

$$= 2.802 \text{ inches.}$$

**10. Mean of Means.** So far we have used subscripts to distinguish between the variates within a set: $x_1, x_2, \cdots, x_N$. By this time the student should be thinking easily in this notation so we may now state an additional use of subscripts. Instead of using $x$ and $y$ to distinguish between two sets of variates we may use $x_1$ and $x_2$. Then to distinguish the variates within a set we would add a second subscript, so for the $x_1$ set the variates are

$$x_{11}, x_{12}, x_{13}, \cdots, x_{1n_1}$$

and for the $x_2$ set the variates are

$$x_{21},\ x_{22},\ x_{23},\ \cdots,\ x_{2n_2}.$$

These are read "$x$ two one," etc., not "$x$ twenty-one," etc. In the notation dealing with one set, $x$ was a variable but $x_1$, $x_2$, etc., were constants. Now $x_1$ and $x_2$ are variables and $x_{11}, x_{12}, \cdots, x_{21}, x_{22}, \cdots$, etc., are constants. Thus $x_1$ and $x_2$ may denote the grades of two sections of mathematics in which there are $n_1$ and $n_2$ students respectively. Then the mean of the first set is

$$(a) \qquad \bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_{1i}$$

and the mean of the second set is

$$(b) \qquad \bar{x}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} x_{2i}.$$

We will now state a useful theorem.

**Theorem VIII.** *If the mean of a set of $n_1$ variates is $\bar{x}_1$ and the mean of another set of $n_2$ variates is $\bar{x}_2$, the mean $\bar{x}$ of the combined sets is*

$$(7) \qquad \bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{N}$$

*where $N = n_1 + n_2$.*

*Proof:* It is obvious from equations $(a)$ and $(b)$ that

$$(c) \qquad n_1\bar{x}_1 + n_2\bar{x}_2 = \sum_{1}^{n_1} x_{1i} + \sum_{1}^{n_2} x_{2i}.$$

If $x$ is allowed to stand for $x_1$ and $x_2$ in succession as shown in the table on page 45 then the right member of $(c)$ may be written $\sum_{1}^{n_1+n_2} x_i$ which denotes the sum of all the variates when they are combined into one set. If this latter sum is divided by the total number of variates $N$ the result is, by definition, their mean. Hence

$$\frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{\sum_{1}^{n_1} x_{1i} + \sum_{1}^{n_2} x_{2i}}{n_1 + n_2} = \frac{\sum_{1}^{n_1+n_2} x_i}{N} = \bar{x}.$$

We may express (7) in more compact notation as follows.

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{2} n_i\bar{x}_i,\ N = \sum_{i=1}^{2} n_i.$$

| | | |
|---|---|---|
| 1 | $x_{11}$ | $x_1$ |
| 2 | $x_{12}$ | $x_2$ |
| 3 | $x_{13}$ | $x_3$ |
| . | . $\}\ \bar{x}_1$ | . |
| . | . | . |
| . | . | . |
| $n_1$ | $x_{1n_1}$ | $x_{n_1}$ $\}\ \bar{x}$ |
| $n_1 + 1$ | $x_{21}$ | $x_{n_1+1}$ |
| $n_1 + 2$ | $x_{22}$ | $x_{n_1+2}$ |
| $n_1 + 3$ | $x_{23}$ | $x_{n_1+3}$ |
| . | . $\}\ \bar{x}_2$ | . |
| . | . | . |
| . | . | . |
| $n_1 + n_2$ | $x_{2n_2}$ | $x_{n_1+n_2}$ |
| | $\sum_{i=1}^{n_1} x_{1i} + \sum_{i=1}^{n_2} x_{2i}$ | $\sum_{i=1}^{n_1+n_2} x_i$ |

This form lends itself to a generalization for $k$ sets so we have the following theorem.

**Theorem IX.** *The mean of a set of $N$ variates which is composed of $k$ subsets is*

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{k} n_i \bar{x}_i \tag{8}$$

where $\bar{x}_i$ is the mean and $n_i$ is the frequency in the $i$th subset and $N = \sum_{i=1}^{k} n_i$.

COROLLARY. *If $n_i = n$ is the same for all the sets, then $N = kn$ and* (8) *reduces to*

$$\bar{x} = \frac{1}{k}\sum_{1}^{k} \bar{x}_i. \tag{9}$$

## Exercises

1. (*a*) Use (1), §3, to find the mean of the following numbers: 18, 42, 23, 16, 103, 61, 49, 95, 113, 10.
   (*b*) For the numbers in (*a*) verify that the sum of their deviations from their mean is zero. . What theorem does this exercise illustrate?
2. Find the deviations of the numbers in Ex. 1 from 50 and verify that the mean of these deviations added algebraically to 50 gives the mean of the numbers themselves.
3. Prove: The sum of the deviations of the variates from their mean is zero.
4. Derive the relation $\bar{x} = c\bar{u} + x_0$.

5. Find the arithmetic mean of the weights of 1000 students given in Table 12. Use (5). *Ans.* 138.65 lbs.
6. Find the mean monthly rainfall at Des Moines from 1890 to 1925, using the frequency distribution which you previously made. *Ans.* 2.55 inches.
7. Find the mean of the distribution of discrete variates given in Table 11.
8. Prove the following theorem: *The mean of a set of variates is unchanged if each variate is replaced by the mean of all the variates.*
9. (*a*) Prove expressions (8) and (9).
   (*b*) The mean grade of one class of 20 students is 76% and of another class of 15 students is 80%. Find the mean of the two classes.
10. The record of freshman scholastic averages for a semester at a certain university were given as follows:

| | $n_i$ | $x_i$ |
|---|---|---|
| Men | 501 | 3.550 |
| Women | 356 | 3.639 |

Find the mean grade for the entire class.

11. Assume that the following fictitious data represent the earnings per week of a certain type of machine shop labor in Illinois establishments:

| *Wage Group* | | *Frequency* |
|---|---|---|
| \$00.0 *under* | \$10.0 | 50 |
| 10.0 | 20.0 | 150 |
| 20.0 | 30.0 | 400 |
| 30.0 | 40.0 | 200 |
| 40.0 | 50.0 | 160 |
| * | | |
| 60.0 | 80.0 | 40 |
| Total | | 1000 |

*Class omitted. Note the different interval in the last class.

The average earnings per week for this same type of labor in all other states of the United States where 9000 men are employed, not counting those in Illinois, are \$30.00 per week.

Compute the arithmetic mean wage (*a*) for Illinois, (*b*) for the entire United States.

Recompute the mean wage for Illinois in such a manner as to check, in the quickest and surest way, the accuracy of the result found in (*a*) above.

12. Find the mean of the following distribution:

| $x$ | $f$ |
|---|---|
| 47.5 | 7 |
| 48.1 | 17 |
| 45.9 | 46 |
| 44.0 | 44 |
| 40.7 | 54 |
| 41.6 | 43 |
| 38.0 | 35 |
| 33.2 | 14 |

**11. The Mode.** That value of the variable which occurs most frequently is called the *mode*. Its chief service is in characterizing a type and it is the kind of average meant by such a phrase as the "average man." There is some difficulty in giving a precise definition of the mode without more advanced mathematics. However, we may say that for a given grouping an approximate value, which we will call the *empirical* mode, is given by the class mark having the largest frequency.[1] Thus, in Table 17 the empirical mode is 1.745 inches.

**12. The Median.** Instead of finding the mean, suppose the $N$ variates are arranged in the order of their magnitude. The median is defined as the value which is greater than half the variates and less than the other half. A more precise definition is as follows:

Let $x_1, x_2, \cdots, x_N$ be a set of real numbers, which may or may not be all different and suppose they are arranged in order of magnitude so that

$$x_1 \leqq x_2 \leqq x_3 \leqq \cdots \leqq x_N.$$

Whenever $N$ is odd, $N = 2k - 1$, the median is $x_k$, the middle one of the $x$'s. If $N$ is even, $N = 2k$, the median is not uniquely defined unless $x_k = x_{k+1}$, in which case the median is this common value. Otherwise, the definition is satisfied by any value of $x$ belonging to the interval

$$x_k \leqq x \leqq x_{k+1},$$

and the median is to this extent indeterminate. In this case it is conventional to take

$$\tfrac{1}{2}(x_k + x_{k+1})$$

as the median.

*Example.* Find the median of the following set of numbers: 10, 6, 5, 25, 15, 18, 20.

Arranging them in order of magnitude we find the median to be 15 (the mean is 14.14). If we add another value, 37, to make $N$ even, the median is $\frac{1}{2}(15 + 18) = 16.5$ (the mean is 17).

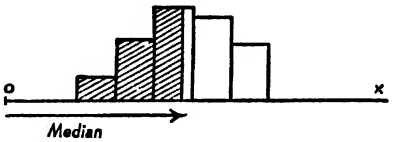
**13. Median of a Frequency Distribution.** *Case I.* For a frequency distribution of *continuous* variates, the median is defined as follows:

DEFINITION: *The median is the value of $x$ for which cum $f = N/2$.* Given such a frequency distribution we may therefore find its

[1] Another method of computing the mode will be given in a later section.

median by forming a cumulative frequency table and interpolating in the *end-x* column for the value of $x$ corresponding to $N/2$.

The method should be clear from the following illustration.




Find the median for the data of Table 2.

| *Interval* | *f* | *End-x* | *Cum f* |
|---|---|---|---|
| | | 29.5 | 0 |
| 30–39 | 2 | | |
| | | 39.5 | 2 |
| 40–49 | 3 | | |
| | | 49.5 | 5 |
| 50–59 | 11 | | |
| | | 59.5 | 16 |
| 60–69 | 20 | | |
| | | 69.5 | 36 |
| 70–79 | 32 | ← Md | ← 50 |
| | | 79.5 | 68 |
| 80–89 | 25 | | |
| | | 89.5 | 93 |
| 90–99 | 7 | | |
| | | 99.5 | 100 |

Here, $N/2 = 50$. This value of *cum f* corresponds to a value of $x$ in the interval 69.5–79.5. Therefore the median is 69.5 plus a fraction of the distance from 69.5 to 79.5. Thus,

| *End-x* | *Cum f* |
|---|---|
| $D_1\left[\begin{array}{c} d_1\left[\begin{array}{c}69.5\\ \text{Median}\\ 79.5\end{array}\right. \end{array}\right.$ | $\left.\begin{array}{c}36\\ 50\\ 68\end{array}\right]d_2\Bigg]D_2$ |

Assuming that the items in any class interval are uniformly distributed over that interval, it follows that the partial differences are proportional to the total differences: $d_1/D_1 = d_2/D_2$. That is,

$$\frac{\text{Median} - 69.5}{79.5 - 69.5} = \frac{50 - 36}{68 - 36}$$

whence,

$$\text{Median} = 69.5 + 10\left(\frac{14}{32}\right)$$
$$= 69.5 + 4.4 = 73.9.$$

This is called "straight line interpolation" or "interpolation by proportional parts." The reason for these names is made clear in the following diagram.

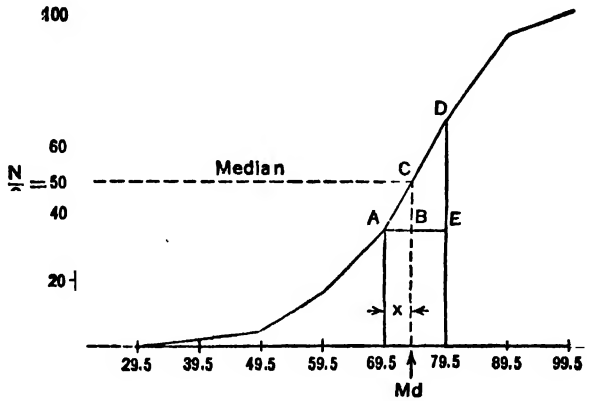



FIG. 8

$\triangle ABC$ is similar to $\triangle AED$

$$\therefore \frac{AB}{AE} = \frac{BC}{ED}$$

$$x = AB = \frac{AE \cdot BC}{ED}$$

$$= \frac{10(50 - 36)}{68 - 36}$$

$$= 10\left(\frac{14}{32}\right)$$

$$= 4.4$$

$$\therefore \text{Md} = 69.5 + x = 73.9.$$

The following formula may also be used to compute the median:

$$Md = {}_{c}x_m + \left[\frac{N}{2} - {}_{b}f_m\right]\frac{c}{f_c}$$

where ${}_cx_m$ is the lower end-value of the median class, $N$ is the total frequency, ${}_bf_m$ the number of variates below the median class, $c$ the class interval, and $f_c$ the frequency of the median class.

*Case II.* In the case of a set of discrete variates there may be no value in the set such that the number of variates which are larger than it is equal to the number less than it. Thus in Table 11 the values of $x$ are integers and 35% of the throws yielded 5 or fewer successes and 65% yielded 6 or more successes. Neither $x = 5$ nor $x = 6$, nor any integer, will exactly split the total frequency into two equal parts. Of course a formal application of the definition given in Case I will give a value of $x$ for which *cum f* is $N/2$. The difficulty is not so much in the interpretation of the fractional result because the same objection could be cited against the mean. But the real difficulty lies in explaining interpolation in a discontinuous function. We cannot assume that the given frequencies are distributed over the interval from one value of $x$ to the next. Perhaps the best we can do in such cases is to make a statement similar to the one above for Table 11. At least such a statement serves to summarize the situation without artificiality.

**14. Graphical Interpretation of Mean, Median, and Mode.** The mean corresponds to the abscissa of the point known in mechanics as the centroid of area. If a thin, homogeneous plate of metal cut in

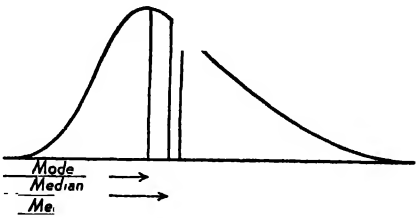



FIG. 9

the shape of a histogram is supported loosely on a horizontal axis through its centroid, the plate will have no tendency to rotate, whatever horizontal direction this axis may assume.

The median of a frequency distribution is the abscissa of a point through which a vertical line will divide the total area of the histogram into two equal parts.

If a distribution could be represented by a smooth curve, then the mode is the abscissa of the highest point on the curve.

Figure 9 shows the position of the three averages in a moderately

skew distribution. If the distribution were perfectly symmetrical then all three of these measures of location would coincide.

There is an interesting empirical relationship between the three quantities which appears to hold for unimodal curves of moderate asymmetry, namely,

$$\text{mean} - \text{mode} = 3(\text{mean} - \text{median}).$$

It is a useful mnemonic to observe that the mean, median, and mode occur in the same order (or reverse order) as in the dictionary; and that the median is nearer to the mean than to the mode, just as the corresponding words are nearer together in the dictionary.[1]

**15. Discussion.** The student primarily interested in the use of these averages in practical statistics might reasonably inquire, "Which of the three averages mentioned should be used in a given problem?" The answer depends upon certain properties peculiar to each average and upon the nature of the data to be averaged.

In most cases the mean is a distinctly superior average. It is rigorously defined, easily computed, and is most tractable in theoretical discussions.

When the median differs considerably from the mean it is likely that the median is the more typical value. The advantage of the median over the mean is recognized in at least three situations:

(*a*) When occasional and unexpected values occur at the ends of the distribution. In such cases the mean would tend to distort the true representation of the typical value, being unduly influenced by the exceptional values.

(*b*) When the data are presented in a table left open at one or both ends. For example, suppose the registrar's office of a university reports the following distribution of grades as given in all departments for a semester:

| *Below* 60 | 60–69 | 70–79 | 80–89 | 90–100 |
|---|---|---|---|---|
| 215 | 1060 | 2217 | 1242 | 506 |

A *cum f* table may be formed and hence the median can be found without any more information about the values less than 60.

(*c*) When the observations cannot be measured numerically but can be ordered.

The mode is best adapted to situations where the word "usual" would be appropriate. Unless a large number of items are con-

[1] M. G. Kendall — *The Advanced Theory of Statistics*, vol. I, p. 35. Lippincott.

sidered the mode can have little practical meaning. It is the appropriate average in certain questions of marketing because manufacturers are interested in the type or quality which is usually in demand. Or again, in an investigation concerning wages and cost of living, the mode would reflect the average situation. Also, in a mathematical treatment of frequency curves the concept of the mode is very useful.

Sometimes a distribution has more than one mode, although this is usually due to heterogeneous material. In this course we will be concerned only with unimodal distributions.

The above remarks about the appropriateness of various averages are made from the standpoint of describing and condensing the data *per se*. A few remarks from a different point of view should perhaps be added here. In the theory of sampling, which deals to a large extent with estimating from a sample certain constants in the parent universe, it is shown that the mean has definite advantages. The mean is much more efficient [1] than the median, for example, in estimating the corresponding average in the universe (except in a special case when the universe is an unusual type).

For a more complete treatment of the applicability of these three averages, the student is referred to the following books:

1. *Theory of Statistics* — Yule and Kendall, Ch. VII.
2. *The Mathematics of Statistics* — Burgess, Ch. V.
3. *Mathematical Statistics* — Camp, p. 40.

### Exercises

1. State what the empirical mode is in each of Tables 8 to 13.
2. Explain why the median is found from interpolating in the *end-x* column and not the *mid-x* column.
3. Read one or more of the references in §15 and write an essay on the advantages and limitations of the mean, median, and mode.
4. Find the median IQ for the data in Table 7.
5. Find the median for the data in Table 9.

**16. Geometric Mean.** The geometric mean of a set of $N$ positive values is the $N$th root of their product. Thus, the geometric mean (G.M.) of two values is the square root of their product, of three values the cube root of their product, and in general for the $N$ values $y_1, y_2, \cdots, y_N$,

$$\text{G.M.} = [y_1 \cdot y_2 \cdot y_3 \cdots y_N]^{\frac{1}{N}}. \tag{10}$$

[1] See *Economic Control of Manufactured Products* — W. A. Shewhart, p. 280. D. Van Nostrand Co.

Equation (10) lends itself to the use of logarithms and frequently they greatly facilitate the computation of G.M. From (10) we have

$$\log \text{G.M.} = \frac{1}{N}[\log y_1 + \log y_2 + \cdots + \log y_N]. \tag{11}$$

Therefore the arithmetic mean of the logarithms of a set of values is the same as the logarithm of the geometric mean of the values themselves.

*Examples:* Find the geometric mean of

(*a*) 3, 6, 12, 24, 48.

*Solution:*

$$\text{G.M.} = [(3^5)(2^{10})]^{1/5} = (3)(2^2) = 12.$$

(*b*) 7.96, 13.82, 22.95, 35.34.

*Solution:*

$$\begin{aligned} \log 7.96 &= 0.90091 \\ \log 13.82 &= 1.14051 \\ \log 22.95 &= 1.36078 \\ \log 35.31 &= 1.54827 \\ 4\,|\,4.95047 & \\ \log \text{G.M.} &= 1.23762 \\ \text{G.M.} &= 17.28 \end{aligned}$$

The geometric mean is the appropriate average when the data are limited at one end of the range and unlimited at the other, and there tends to be a constant rate of change from one $y$ value to the next. This is characteristic of values which tend to form a geometric progression, *i.e.*, which tend to follow the simple exponential law

$$y = ar^x. \tag{12}$$

The student will recall from algebra that a geometric progression can be put in the form

| $x$ | 0 | 1 | 2 | $\cdots$ | $x$ |
|---|---|---|---|---|---|
| $y$ | $a$ | $ar$ | $ar^2$ | $\cdots$ | $ar^x$ |

The value of any term in the $y$ series is a function of the exponent of $r$ since $a$ and $r$ are constants. The functional relationship is therefore represented by (12).

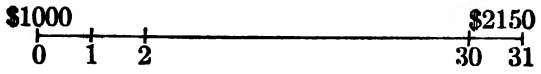
The growth of many quantities in nature follows this law and it is sometimes called the law of natural growth. With $x$ referring to

time, $y$ may represent, for example, the population of a city, the enrollment of a school, the weight of a quantity, or the number of bacteria in a culture. The accumulated amount $S$, of $P$ dollars invested at $i$ rate of interest, compounded periodically for $n$ periods also takes the form of (12), namely,

$$S = P(1 + i)^n,$$

where $r$ is now $(1 + i)$, $a$ is $P$, and $n$ and $S$ are the variables corresponding to $x$ and $y$.

Thus, if \$1000 increased at compound interest to \$2150 in 31 years,




the geometric average rate at which the money increased is found as follows

$$r^{31} = (1 + i)^{31} = \frac{2150}{1000}$$
$$1 + i = (2.15)^{1/31}$$
$$= 1.025$$
$$i = 2\tfrac{1}{2}\%.$$

Since there was an increase of $\frac{1150}{1000} = 115\%$, the arithmetic average would be $\frac{115}{31} = 3.7\%$ which is also the simple interest rate.

If $y$ in equation (12) represents population, and we are given two values of $y$ corresponding to two dates $N$ years apart, the geometric mean enables us to find a fairer estimate of the value of $y$ at the mid date than would be given by any other average. For example, suppose we are given that the population of a city was 2500 in 1920 and 5000 in 1930. We wish to estimate the population in 1925 and to find the average annual rate of increase. If we are given no other information, our best estimate for 1925 is given by

$$\text{G.M.} = (y_1 \cdot y_2)^{1/2} = (2500 \times 5000)^{1/2} = 3535.$$

The average annual rate of increase is obtained by solving (12) for $r$ as follows:

$$5000 = 2500r^{10}$$
$$2 = r^{10}.$$

Hence $r = \sqrt[10]{2} = 1.0718 = 107.18\%$, so that the average annual rate of increase is 7.18%. It is now possible to estimate the population for *any* intermediate year. Thus, for 1928, we have from (12):

$$y = 2500(1.0718)^8 = 4353.$$

The geometric mean is also used in economics in averaging " index numbers " which are essentially the ratios of prices of commodities at one date to their prices at another date. In general it is the appropriate average when emphasis is on the rate or percentage of change rather than the amount.

**17. Harmonic Mean.** Another average which has long been known and which is required in certain problems is the *harmonic mean* (H.M.). For the $N$ positive values $x_1, x_2, \cdots, x_N$, it is defined as the reciprocal of the arithmetic mean of the reciprocals of the values. In symbols,

$$\text{(13)} \qquad \text{H.M.} = \frac{1}{\frac{1}{N}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_N}\right)} = \frac{N}{\sum\left(\frac{1}{x_i}\right)}.$$

This measure is used in averaging ratios, such as rates and prices, when certain conditions are agreed upon.

In the case of time rates, we have ratios between two quantities one of which is in units of time, which we will denote by $t$, and the other is in units of some element like distance or accomplishment or temperature, etc. Denote this second element, different from time, by $d$. Then we make the following observations:

(*a*) A rate may be stated either in the form $d/t$ or in the form $t/d$. Thus, a car which travels at the rate of 30 miles per hour may also be said to travel at the rate of 2 minutes per mile. In this illustration the second form is not the usual way of expressing the rate, but there are cases in which the form $t/d$ is usual. When we say a man takes 10 seconds to run 100 yards we are expressing his rate in time per unit of distance ($t/d$).

(*b*) In averaging rates one should first decide whether $d$ or $t$ should properly be the basic or " fixed " element in the discussion. Occasionally there is a difference of opinion about which element should most appropriately be regarded as fixed. For example, suppose a class of students has been given 15 minutes in which to work as many as they can of a given list of problems, and the number of problems

worked correctly by each student recorded. Some educational statisticians would say that *time* should be the fixed element here and that number of problems solved (in a unit of time) should be the variable. Others would say that the number of minutes ($t$) a student required to work one problem is the proper variable and that a problem ($d$) should be regarded as the fixed element in the discussion.

In one case the rates are equally weighted in the sense of time and in the other case they are equally weighted in the sense of the element $d$.

(*c*) The harmonic mean of the rates expressed in the form $d/t$ gives the same result as the arithmetic mean of the same rates expressed in the form $t/d$. This is evident from equation (13) if it is written in the form,

$$\frac{1}{\text{H.M.}} = \frac{1}{N}\sum\frac{1}{x_i}$$

and from the fact that rates in one form are merely the reciprocals of the same rates in the other form.

As an illustration, let us consider three cars:

I
A travels at the rate of 15 miles per hour ($\frac{1}{4}$ mile per minute),
B travels at the rate of 20 miles per hour ($\frac{1}{3}$ mile per minute),
C travels at the rate of 30 miles per hour ($\frac{1}{2}$ mile per minute).

But their rates could just as well have been stated as

II
A travels at the rate of 4 minutes to the mile,
B travels at the rate of 3 minutes to the mile,
C travels at the rate of 2 minutes to the mile.

The harmonic mean of the rates as stated in I is 20 miles per hour; *i.e.*, $\frac{1}{3}$ of a mile per minute, and the arithmetic mean of the rates as stated in II is 3 minutes per mile or again, 20 miles per hour. (Verify.)

The third observation, *i.e.*, (*c*) above, suggests the following discussion. The *arithmetic* mean of the rates in I is $21\frac{2}{3}$ m.p.h. and this is the harmonic mean of the rates as stated in II.

The question arises, which is the correct average, 20 m.p.h. or $21\frac{2}{3}$ m.p.h.? The problem is indeterminate until it is agreed whether time or distance is the fixed element. The correct average will differ

according to the condition agreed upon. This will be made clear in the following analysis.

*Case I.* Let

$$x_i = \frac{d_i}{t_i}$$

denote the $i$th rate, $i = 1, 2, \cdots, n$. Then the average rate is

$$\frac{D = \text{total distance}}{T = \text{total time}} = \frac{t_1x_1 + t_2x_2 + \cdots + t_nx_n}{t_1 + t_2 + \cdots + t_n}$$

*Condition* 1. Let distance be the fixed element, *i.e.*, let $d$ be constant. Then $d = t_ix_i$, and $t_i = d/x_i$. Therefore, the expression for average rate becomes

$$\frac{\sum t_ix_i}{\sum \frac{d}{x_i}} = \frac{nd}{d\sum \frac{1}{x_i}} = \frac{1}{\frac{1}{n}\sum \frac{1}{x_i}}$$

which is the harmonic mean.

*Condition* 2. Suppose $t$ is the fixed element. Then $\sum t_ix_i$ becomes $t\sum x_i$ since $t$ is a constant, and $\sum t$ becomes $nt$. Hence, we have for the average rate,

$$\frac{D}{T} = \frac{t\sum x_i}{nt} = \frac{1}{n}\sum x_i$$

which is the arithmetic mean.

*Case II.* Let $x_i = t_i/d_i$ denote the $i$th rate. Then the average rate is

$$\frac{T = \text{total time}}{D = \text{total distance}} = \frac{\sum t_i}{\sum d_i}.$$

*Condition* 1. Suppose $d$ is the fixed element. Then $t_i = dx_i$ and $d = t_i/x_i$. Hence, we have

$$\frac{T}{D} = \frac{d\sum x_i}{nd} = \frac{\sum x_i}{n}.$$

*Condition* 2. Let $t$ be fixed. Then $d_i = t/x_i$ and the average rate is

$$\frac{T}{D} = \frac{nt}{t\sum \frac{1}{x_i}} = \frac{1}{\frac{1}{n}\sum \frac{1}{x_i}}.$$

We therefore state the following rules for averaging rates:

*Rule* 1. The harmonic mean is used whenever the fixed element is $d$ and the rates are expressed in the form $d/t$, or when the fixed element is $t$ and the rates are expressed in the form $t/d$.

*Rule* 2. The arithmetic mean should be used when the fixed element is $t$ and the rates are expressed in the form $d/t$, or when the fixed element is $d$ and the rates are expressed in the form $t/d$.

In the case of prices, which are of course ratios, a similar discussion holds except that now the unit of time is to be replaced by a unit of money. Therefore, prices are ratios between two quantities, one of which is in units of money and the other in units of some commodity or service. They may be stated as so much money per unit of commodity ($m/c$), or as so many units of commodity per dollar ($c/m$). Thus, if 100 bushels of wheat are exchanged for 75 dollars of gold, the price of the wheat in terms of gold is $75 \div 100$, or three-fourths of a dollar of gold per bushel of wheat. Contrariwise, the price of gold in terms of wheat is $100 \div 75$, or one and one-third bushels of wheat per dollar of gold. Thus, there are always two prices in any exchange.

The correct average will depend upon how the prices are stated and upon whether a unit of the commodity (or service) or a unit of money is the fixed element.

The following papers in *The Journal of the American Statistical Association* are recommended:

1. "The Nature and Use of the Harmonic Mean" — W. F. Ferger, vol. 26 (1931), pp. 36–40.
2. "Calculating the Geometric Mean from a Large Amount of Data" — Zenon Szatrowski, vol. 41 (1946), pp. 218–220.

## Examples

1. In a certain factory a unit of work is completed by A in four minutes, by B in five minutes, by C in six minutes, by D in ten minutes, and by E in twelve minutes. What is their average rate of working? At this rate how many units will they complete in a six-hour day?

   *Solution.* The rates are expressed in the form $t/d$ but it would seem appropriate to regard $t$ as the basic or fixed element since output per unit of *time* appears to be the important consideration here. So by Rule 1,

$$\text{H.M.} = \frac{5}{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{12}},$$

that is,

$$\text{H.M.} = \frac{300}{48} = 6\tfrac{1}{4} \text{ minutes per unit.}$$

In 360 minutes they will complete $\frac{4(360)5}{25} = 288$ units.

**2.** A tourist purchases gasoline at three stations, as follows:

| *Station* | *Number of gallons of gasoline for* \$1.00 |
|---|---|
| A | 5 |
| B | 7 |
| C | 6 |

Here the prices are given in the form $c/m$ and it would seem appropriate to regard gallon ($c$) as the fixed element and prices ($m$) per gallon as the variable quantities which are to be averaged. Hence, replacing $d/t$ by $c/m$ and " rates " by " prices " in Rule 1, we are led to find the harmonic mean.

$$\therefore \text{H.M.} = \frac{3}{\frac{1}{5} + \frac{1}{6} + \frac{1}{7}}$$

$$= \frac{630}{107} \text{ gals. per \$1.00}$$

$$= \frac{\$107}{630} \text{ per gal.}$$

## Exercises

1. (*a*) The arithmetic mean of a set of 30 numbers is 82. What is the sum of these numbers?
   (*b*) The G.M. of ten numbers is 1.40. What is the product of these ten numbers?
2. In chemistry a student was graded 65 in final examination, 85 in recitation and 80 in laboratory. These grades were weighted 1, 2, and 3 respectively. Find the student's average grade.
3. At the end of his first semester in college a freshman had credits as follows: 4 hours of mathematics with a grade of 88, 4 hours of English with a grade of 80, 3 hours of history with a grade of 85, and 4 hours of physics with a grade of 78. What was his average grade per hour of credit?
4. Find the median of Table 12.
5. The population of a city increased in 5 years from 225,000 to 245,000. What was the average increase per year? What was the average annual rate of increase?
6. The number of bacteria in a certain culture was found to be $4 \times 10^6$ at noon of one day. At noon the next day the number was found to be $9 \times 10^6$. If the number increased at a constant rate per hour, how many bacteria were there at midnight?

7. Find the average (G.M.) rate of interest for five years during which the interest rates were 4.25%, 5.3%, 4.65%, 3.86%, 4.38%.
   *Hint.* $(1 + i)^5 = (1.0425)(1.053)(1.0465)(1.0386)(1.0438)$.
8. Find the harmonic mean of the first fifteen positive integers.
9. For two positive numbers, $a$ and $b$, the geometric mean is $x = \sqrt{ab}$. This is also called the mean proportional between $a$ and $b$, since $a : x = x : b$. By drawing a semicircle on $a + b$ as diameter, show how the value of $x$ can be constructed geometrically.
10. The following table gives the population of the U. S. at each 10-year census from 1860 to 1920.

| *Year* $x$ | *Population (millions)* | *Ratio of Each Census Figure to Preceding* |
|---|---|---|
| 1860 | 31.4 | |
| 70 | 38.6 | 1.23 |
| 80 | 50.2 | 1.30 |
| 90 | 63.0 | 1.25 |
| 1900 | 76.0 | 1.20 |
| 10 | 92.0 | 1.21 |
| 20 | 105.7 | 1.15 |

What is the average rate of increase per decade? Using this average, estimate the population for 1930 from the 1920 census figure.

11. If a series of positive variates form a geometric progression show that their logarithms form an arithmetic progression.
12. Find the geometric mean of the following:
    (*a*) 2, 4, 8, 16, 32.
    (*b*) 47, 92, 123, 218.
13. Given two sets of $n$ positive variates each:

$$x_{11}, x_{12}, x_{13}, \cdots, x_{1n}$$
$$x_{21}, x_{22}, x_{23}, \cdots, x_{2n}.$$

Prove that the geometric mean of the ratios of corresponding variates in the two sets is equal to the ratio of their geometric means.

14. (*a*) For a frequency distribution of positive variates show that (10) becomes

$$\text{G.M.} = \left[ x_1^{f_1} \cdot x_2^{f_2} \cdots x_k^{f_k} \right]^{\frac{1}{N}}$$

where $k$ is the number of different values of $x$ in the set, any exponent $f_i$ is the number of times $x_i$ is repeated, and $N = \sum_1^k f_i$.

(*b*) What is the expression for log G.M. when G.M. is defined as in (*a*)?

15. A wholesale firm has twelve travelling salesmen who make trips of essentially the same length. Of these, eight make their trip in 20 days and four in 15 days. What is the average time per trip? *Ans.* 18 days.

**16.** State two rules for averaging prices similar to those given for averaging rates. Give illustrations.

**17.** Consider any two positive variates $x_1$ and $x_2$. Prove that their geometric mean is equal to the geometric mean between their arithmetic mean and their harmonic mean.

**18.** (*Burgess*) The following problem arose in a statistical office in Washington during World War I: Suppose 20 boats make 6 trans-atlantic trips each per year, giving as the time for a "turn around" (*i.e.*, time between consecutive departures from the same ports), one-sixth year — approximately 60 days, and that 10 boats make 4 trips per year, giving as their time for a "turn around" one-fourth year, approximately 90 days. (A year of 360 days is used merely for convenience.) What is the average number of days per turn around?

*Hint.* If we think of the rates expressed as "trips per year" then $x = d/t$. If $t$ is regarded as the fixed element, then by Rule 2 the arithmetic mean is indicated, and $x = 6$ for 20 values of $x$, and $x = 4$ for 10 values.

If we think of the rates expressed as "days per trip" then $x = t/d$. If $t$ is the fixed element, by Rule 1 the harmonic mean is the correct average, and $x = 60$ for 20 values and $x = 90$ for 10 values. *Ans.* $5\frac{1}{3}$ trips per year or 67.5 days per trip.

**19.** Show that if $2a$ is the harmonic mean of the two rational numbers $b$ and $c$, then the sum of the squares of the three numbers $a$, $b$, and $c$ is the square of a rational number.

(Reference: *American Mathematical Monthly*, June 1935, p. 394.)

**20.** (*a*) If A, G, and H represent, respectively, the arithmetic, geometric, and harmonic means of $N$ unequal positive variates, prove that

$$H < G < A$$

(Reference: Burgess' text, p. 101.)

(*b*) What can you say if the $N$ positive variates are equal?

**21.** A plane travels one half of a given distance $D$ in miles at a speed of $x_1$ miles per hour, and the remaining half distance at a speed of $x_2$ miles per hour. Show that the average speed for the entire distance is the harmonic mean of $x_1$ and $x_2$. Half of this average speed is called the "radius of action per hour"; *i.e.*, it is the outbound distance that a plane can travel and return in one hour. The "radius of action" of a plane would be the "radius of action per hour" multiplied by the number of hours in flight.

# CHAPTER IV

## MOMENTS

**1. Moments about an Arbitrary Origin.** One of the general problems of statistics is to summarize and characterize data. In the words of R. A. Fisher,

> A quantity of data which by its mere bulk may be incapable of entering the mind is to be replaced by relatively few quantities which shall adequately represent the whole, or which, in other words, shall contain as much as possible, ideally the whole, of the relevant information contained in the original data.[1]

These " relatively few quantities " are usually expressed in terms of moments. Moments are of different orders and the student is already familiar with what is now to be known as the first moment, namely, the arithmetic mean of the first powers of the variates. We will also need in our work the arithmetic means, respectively, of the second, third, and fourth powers of the variates. With reference to an arbitrary origin, moments are denoted by $\nu$ (the Greek letter nu) with a subscript specifying the order.

The first four moments, relative to the $x$-origin and in the $x$ unit, are defined as follows:

$$(1) \qquad \begin{cases} \nu_1 = \dfrac{1}{N}\sum f_i x_i = \bar{x} \\ \nu_2 = \dfrac{1}{N}\sum f_i x_i^2 \\ \nu_3 = \dfrac{1}{N}\sum f_i x_i^3 \\ \nu_4 = \dfrac{1}{N}\sum f_i x_i^4, \end{cases}$$

$i$ varying from 1 to $k$.

A more general definition of the $\nu$'s is

$$(1a) \qquad \nu_r = \frac{1}{N}\sum_1^k f_i(x_i - x_0)^r$$

[1] *Foundations of Theoretical Statistics*, Philosophical Transactions of the Royal Society, vol. 222A (1922), p. 309.

for the $r$th moment about an arbitrary point $x_0$. When $x_0 = 0$ and $r = 1, 2, 3$, and 4, we have the definitions stated in (1). If $r = 0$ we have the zero*th* moment and $\nu_0 = 1$.

In statistics we work with moments per unit frequency. The term " moment " has its origin in mechanics where we speak of the " moment of a force." Suppose we have a rigid bar, called a lever, with one point of support known as a fulcrum (Figure 10). If a force $f_1$ is applied to the lever at a distance $x_1$ from the fulcrum $O$, the product $x_1 f_1$ is called the moment of the force. If there are two or more such forces $f_1$, $f_2, \cdots, f_k$, acting in the same direction, and at the distances $x_1, x_2, \cdots, x_k$, respectively from $O$, the total moment of all these forces is

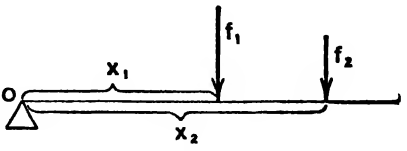


Fig. 10

$$f_1x_1 + f_2x_2 + \cdots + f_kx_k = \sum f_ix_i.$$

If the distances $x$ are squared, we have $\sum f_ix_i^2$ as the total second moment, and $\sum f_ix_i^r$ represents the $r$th moment.

It is by analogy with this mechanical concept that the expressions in (1) are called statistical moments (per unit frequency) about zero as origin.

## Exercises

1. Write out the expanded form of the $\nu$'s defined in (1).
2. Calculate the values of $\nu_1$, $\nu_2$, and $\nu_3$ for the following distributions:

(*a*)

| $x$ | $f$ |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |
| 3 | 10 |
| 4 | 5 |
| 5 | 2 |

(*b*)

| $x$ | $f$ |
|---|---|
| −3 | 1 |
| −2 | 3 |
| −1 | 5 |
| 0 | 5 |
| 1 | 3 |
| 2 | 1 |

3. (*a*) Prove that $\nu_0$ is always equal to unity.
   (*b*) Prove that moments of even order are always positive or zero, but that moments of odd order may be positive, negative, or zero.

(c) Show that the odd moments are all zero if both the $x$'s and $f$'s are symmetrical with respect to the origin of $x$, as, for example,

| $x$ | −1.5 | −1.0 | −0.5 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|---|---|
| $f$ | 1 | 2 | 3 | 3 | 2 | 1 |

**2. Moments in Units of the Class Interval.** In Chapter III, §8, the mean in the $x$ unit was obtained by first finding the mean in the $u$ unit, viz., $\frac{1}{N}\sum f_i u_i$, and then changing over into the $x$ unit by multiplying by the interval $c$. In our subsequent work, which requires the higher moments, we shall find it convenient to use a similar procedure, and find those moments in the $u$ unit, where $u = (x - x_0)/c$. It is desirable, therefore, in labeling the moments for any distribution, to specify whether they are in the unit of $x$ or $u$. This is commonly done by the use of a second subscript on $\nu$. Thus $\nu_{r:u}$ denotes the $r$th moment in the $u$ unit and relative to the $u$-origin. Therefore,

$$(2)\qquad \begin{cases} \nu_{1:u} = \dfrac{1}{N}\sum f_i u_i = \bar{u} \\ \nu_{2:u} = \dfrac{1}{N}\sum f_i u_i^2 \\ \nu_{3:u} = \dfrac{1}{N}\sum f_i u_i^3 \\ \nu_{4:u} = \dfrac{1}{N}\sum f_i u_i^4. \end{cases}$$

Similarly, $\nu_{r:x}$ will mean $\frac{1}{N}\sum f_i x_i^r$. When there is no ambiguity, the second subscript on $\nu$ may be omitted.

**3. Moments about the Mean.** Formulas (1) and (2) define the moments taken about zero as origin although in different units. When the mean is chosen as origin we have the most important set of moments in the theory of statistics. In this case the Greek letter $\mu$ (mu) is used to denote the moments, and it is always understood that the use of $\mu$ specifies the mean as origin. It does not, however, designate the unit, so the second subscript may still be necessary. Therefore, the $r$th moment about the mean is defined by either of the

following expressions:

$$(3)\qquad \begin{cases} \mu_{r:x} = \dfrac{1}{N}\sum f_i(x_i - \bar{x})^r \\ \mu_{r:u} = \dfrac{1}{N}\sum f_i(u_i - \bar{u})^r. \end{cases}$$

The mean is a sort of balance point. If weights proportional to the frequencies are suspended along a horizontal bar at distances from one end proportional to the numbers representing the class marks, then the bar will balance at the weighted mean of the distances. In mechanics this point is known as the abscissa of the *center of gravity* or *centroid*. Theorem VI of Chapter III, §7, is another way of saying that the given distribution is in equilibrium about this point.

**4. Relations between the μ's and ν's.** We shall see that the descriptive constants mentioned at the beginning of the chapter are defined in terms of the moments about the mean, but the moments about an arbitrary point are easier to calculate. In other words, what we desire are the values of $\mu_r$, but their computation directly from the definitions (3) may be very laborious even in the $u$ unit due to the fact that $(u - \bar{u})$ usually involves decimals. Raising these decimals to the second, third, and fourth powers becomes tedious even with the aid of a computing machine. On the other hand, the $\nu$'s defined in (2) are readily computed. Therefore, instead of computing the $\mu$'s directly we obtain them indirectly from the $\nu$'s. The relations between the $\mu$'s and $\nu$'s can be found by expanding, by the Binomial Theorem, either of the expressions following the $\sum$'s in (3) for $r = 2, 3, 4$. This is done in the $u$ unit as follows:

$$\begin{aligned} \mu_2 &= \frac{1}{N}\sum f_i(u_i - \bar{u})^2 \\ &= \frac{1}{N}\sum f_i u_i^2 - 2\bar{u}\cdot\frac{1}{N}\sum f_i u_i + \bar{u}^2 \\ &= \nu_2 - 2\bar{u}\nu_1 + \bar{u}^2 \\ &= \nu_2 - (\nu_1)^2, \text{ since } \bar{u} = \nu_1 \qquad (4) \\ \mu_3 &= \frac{1}{N}\sum f_i(u_i - \bar{u})^3 \\ &= \nu_3 - 3\nu_2\cdot\nu_1 + 2(\nu_1)^3 \qquad (5) \\ \mu_4 &= \nu_4 - 4\nu_3\cdot\nu_1 + 6\nu_2(\nu_1)^2 - 3(\nu_1)^4. \qquad (6) \end{aligned}$$

These formulas are important and the student should be able to derive them. It should be apparent that these moment relations hold also in the $x$ unit. However, if we have the $\mu$'s in the $u$ unit and we desire them in the $x$ unit they may be found as follows:

$$(7) \qquad \begin{cases} \mu_{2:x} = c^2\mu_{2:u} \\ \mu_{3:x} = c^3\mu_{3:u} \\ \mu_{4:x} = c^4\mu_{4:u}. \end{cases}$$

The first of the relations given in (7) is proved below. The others may be proved in a similar manner.

$$\mu_{2:x} = \frac{1}{N}\sum f_i(x_i - \bar{x})^2 \text{ by definition,}$$

$$= \frac{1}{N}\sum f_i(x_0 + cu_i - x_0 - c\bar{u})^2 \text{ by (4}a\text{) and (5), Chapter III,}$$

$$= \frac{c^2}{N}\sum f_i(u_i - \bar{u})^2 = c^2\mu_{2:u}.$$

We see that the indirect method of computing the $\mu$'s (in the $u$ unit) involves two steps. First the $\nu$'s are computed according to the definitions in (2). This step is illustrated in Table 18. Then we calculate the $\mu$'s by substituting the computed $\nu$'s in relations (4), (5), and (6). The $\mu$'s in the $x$ unit could then be obtained, if desired, by means of (7).

Before proceeding with the second step it is desirable to check the $\nu$'s or, at least, the totals of the columns from which they are obtained. This can be done if we have another column headed $f(u + 1)^4$, and observe that

$$\sum f(u + 1)^4 = \sum fu^4 + 4\sum fu^3 + 6\sum fu^2 + 4\sum fu + \sum f.$$

This is known as *Charlier's check.* An alternative one is to check the entries in the column $fu^4$ against the proper entries in Pearson's *Tables for Statisticians and Biometricians*, Table L.

Charlier's check is a necessary but not a sufficient check. That is to say, compensating errors may occur which this check would not detect. However, the occurrence of such errors is very unlikely.

Applying Charlier's check to Table 18 we have

$$1220 = 1088 + 4(-236) + 6(176) + 4(-20) + 100 = 1220.$$

TABLE 18 — MOMENTS FOR DISTRIBUTION OF GRADES

| *Data* | | *Computations* | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $f$ | $u$ | $fu$ | $fu^2$ | $fu^3$ | $fu^4$ | $f(u+1)^4$ |
| 34.5 | 2 | −4 | − 8 | 32 | −128 | 512 | 162 |
| 44.5 | 3 | −3 | − 9 | 27 | − 81 | 243 | 48 |
| 54.5 | 11 | −2 | −22 | 44 | − 88 | 176 | 11 |
| 64.5 | 20 | −1 | −20 | 20 | − 20 | 20 | 0 |
| 74.5 | 32 | 0 | 0 | 0 | 0 | 0 | 32 |
| 84.5 | 25 | 1 | 25 | 25 | 25 | 25 | 400 |
| 94.5 | 7 | 2 | 14 | 28 | 56 | 112 | 567 |
| Sums | 100 | | −20 | 176 | −236 | 1088 | 1220 |
| $\frac{1}{N}$ Sums | 1 | | $-.20$ $\nu_{1:u}$ | 1.76 $\nu_{2:u}$ | −2.36 $\nu_{3:u}$ | 10.88 $\nu_{4:u}$ | For Charlier's check |

Hence we may proceed with confidence to compute the $\mu$'s. Using relations (4), (5), and (6):

$$\mu_{2:u} = 1.76 - (-.20)^2 = 1.72$$

$$\mu_{3:u} = -2.36 - 3(1.76)(-.20) + 2(-.20)^3 = -1.320$$

$$\mu_{4:u} = 10.88 - 4(-2.36)(-.20) + 6(1.76)(-.20)^2 - 3(-.20)^4 = 9.4096.$$

The following check, which can be handled readily on a machine, may be used to check the $\mu$'s:

$$\nu_4 = \frac{1}{N}\sum f_i x_i^4 = \frac{1}{N}\sum f_i[(x_i - \nu_1) + \nu_1]^4$$
$$= \mu_4 + 4\mu_3\nu_1 + 6\mu_2\nu_1^2 + \nu_1^4.$$

Before explaining the applications of $\mu_2$, $\mu_3$, and $\mu_4$ we present some exercises which will aid the student in mastering the procedure thus far developed.

### Exercises

1. (*a*) Verify relations (4), (5), and (6).
   (*b*) Show that these relations hold also in the $x$ unit.
   (*c*) Prove that $\mu_1 = 0$ in any unit.
   (*d*) When $f = 1$, show that

$$\mu_{2:x} = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \bar{x}^2$$

2. Verify the relations given in (7).
3. Using Table 18 as a model find the $\nu$'s for Iowa City rainfall by extending Table 17.
4. Find the $\mu$'s from your results in Exercise 3 above.

**5. Standard Deviation.** Formula (4), $\mu_2 = \nu_2 - \nu_1^2$, is perhaps the most important of the moment relations for elementary statistics. It states that the second moment about the mean is equal to the second moment about zero diminished by the square of the mean measured from zero.

Many of the definitions in statistics are essentially those of physics and mechanics. The analogy between the mean and centroid has been mentioned. The above statement about formula (4) is a well-known proposition in mechanics when the word centroid is substituted for mean.

In mechanics the equivalent of $N\mu_2$ is called the *moment of inertia* (about the axis through the centroid) and $(\mu_2)^{1/2}$ is the *radius of gyration*. These notions are carried over in statistics. Suppose a thin metal plate in the shape of a histogram is rotating about a vertical axis through its centroid. There is a distance from the centroid at which the entire mass of the histogram could be concentrated without changing its moment of inertia. This distance is the square root of $\mu_2$. It is an *average rotational radius* for all particles of the rotating mass. In statistics, $(\mu_2)^{1/2}$ is called the *standard deviation* and is denoted by the small Greek letter $\sigma$. Therefore we have

$$(8) \qquad \begin{cases} \sigma_x = \sqrt{\mu_{2:x}} \\ \sigma_u = \sqrt{\mu_{2:u}} \\ \sigma_x = c\sigma_u. \end{cases}$$

We shall see later that $\sigma$ is a measure of what is called *dispersion*. More precisely, it measures the extent to which the data are spread out " on the average " on either side of the mean. (See Figure 11.) The student will obtain a more complete understanding of $\sigma$ as the course develops.

The mean and standard deviation are always expressed finally in the same units as the variates. If $x$ represents inches, we desire the mean and standard deviation in inches. When obtained they should be labelled appropriately.

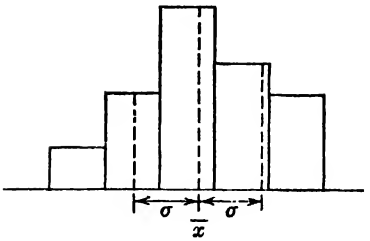



FIG. 11

*Example.* For Table 18, we have

$$\bar{x} = c\bar{u} + x_0 = 10(-.20) + 74.5 = 72.5\%$$
$$\sigma_u = \sqrt{\mu_{2:u}} = (1.72)^{1/2} = 1.31$$
$$\sigma_x = c\sigma_u = 10(1.31) = 13.1\%$$

Thus, we have explained the use of the first and second moments.

The student will observe that the change from $\sigma_u$ to $\sigma_x$ does not involve $x_0$. The standard deviation is affected by the change in units but is independent of the origin of reference. To prove this let $x' = x - x_0$, whence $\bar{x}' = \bar{x} - x_0$ (why?). Then

$$\sigma_{x'}^2 = \mu_{2:x'} = \frac{1}{N}\sum f_i(x_i' - \bar{x}')^2$$
$$= \frac{1}{N}\sum f_i[x_i - x_0 - \bar{x} + x_0]^2$$
$$= \frac{1}{N}\sum f_i(x_i - \bar{x})^2$$
$$= \mu_{2:x} = \sigma_x^2.$$

This suggests the more general

**Theorem.** *The value of $\mu_r$ remains invariant under a transformation which changes only the origin of reference of the variates.*

The student is asked to prove the equivalent of this theorem in Exercise 3 after §9.

**6. Standard Units.** The above section explains $\mu_2$. There remains the explanation of $\mu_3$ and $\mu_4$. We will lead up to this by defining standard units. We have mentioned the transformation $x' = x - \bar{x}$. Another very useful transformation consists in measuring such deviations from the mean in units of the standard deviation,

$\sigma_x$, of the entire distribution. They are then known as standard units and will be designated by $t$. Thus,

$$t = \frac{x - \bar{x}}{\sigma_x} = \frac{x'}{\sigma_x}. \tag{9}$$

Graphically, this translates the origin to the mean and measures distances along the horizontal axis in terms of $\sigma_x$. It is a special case of the more general transformation

$$u = \frac{x - x_0}{c}.$$

The significant characteristic of the $t$ variate is its independence of the unit in which the original measurements were taken. For example, suppose we were concerned with obtaining the linear measurements of a set of individuals. One distribution of variates would result if the measurements were made in feet. In this case $x'$, $\bar{x}$, and $\sigma_x$ would also be in feet. If the measurements were taken in inches, then $x'$, $\bar{x}$, and $\sigma_x$ would be in inches, and each of these values would be, numerically, twelve times as large as the corresponding numbers in the first distribution. However, the variates expressed in standard units would be the same for the two distributions. Thus if

$$\bar{x} = 50 \text{ ft.} = 50(12) \text{ in.},$$

and

$$\sigma_x = 5 \text{ ft.} = 5(12) \text{ in.},$$

then for an individual measurement of $x = 60$ ft. $= 60(12)$ in., we have

$$t = \frac{10 \text{ ft.}}{5 \text{ ft.}} = \frac{10(12) \text{ in.}}{5(12) \text{ in.}}$$

$$t = 2 = 2.$$

It is obvious, therefore, that standard units provide a basis for comparing distributions. Moreover, they make possible important simplifications in certain mathematical operations.

With the aid of a computing machine, a distribution may be easily transformed into standard units by means of the so-called continuous process. To illustrate, suppose for the distribution of Table 9 (§11, Chapter I), it has been found that

$$\bar{x} = 47.712 \text{ lbs.}$$

$$\sigma_x = 5.772 \text{ lbs.}$$

By relation (9), then,

$$t = \frac{x - 47.712}{5.772} = .17325x - 8.2661.$$

Referring to the discussion of the continuous method given in the Introduction, we observe that here $k = -8.2661$, $n = .17325$, and we desire the values of $t$ corresponding to the values of $x$ given in Table 9. For the values of $x$ such that $nx < k$, we write the above relation in the form

$$-t = 8.2661 - .17325x.$$

The procedure[1] now is to register 8.266100 on the product register, punch the constant factor .17325 on the keyboard, and then by turning the crank backward so that the successive values of $x$ appear on the revolution register, we subtract from $k$ the products of this multiplier and the values of $x$. The various values of $x$ are built over from one to another without clearing the dial. The resulting values of $-t$ are read at each stage from the product register until we get $-t = 0.383$. From here, $nx > k$, so we clear the dials and start over using the original form of the relation between $x$ and $t$. We now register $-8.266100$ on the product register by turning the crank backward, punch .17325 on the keyboard, and turn the crank forward to form the values of $x$ on the revolution register. The values of $t$ are read as before from the product register at each stage of the build-over process. In this way the following set of standard variates is obtained:

TABLE 19

| $x$ | $f$ | $t$ |
|---|---|---|
| 29.5 | 1 | −3.155 |
| 33.5 | 14 | −2 462 |
| 37.5 | 56 | −1.770 |
| 41.5 | 172 | −1.076 |
| 45.5 | 245 | −0.383 |
| 49.5 | 263 | 0.310 |
| 53.5 | 156 | 1.003 |
| 57.5 | 67 | 1.696 |
| 61.5 | 23 | 2.389 |
| 65.5 | 3 | 3.082 |

[1] If automatic machines are available the instructor will explain the procedure.

We see from Table 19 that a range of $t = \pm 3$ takes in practically all the variates. This is typical of the more common distributions.

If $\bar{x} = 0$, then $t = x/\sigma$ and the origin of $t$ is the same as the origin of $x$. Some writers use $X$ to denote the variates (*i.e.*, pounds, dollars, temperatures, etc.), and use $x$ to denote deviations from the mean. In that notation, $t = x/\sigma$ would have the same meaning as our equation (9). Occasionally in later chapters we shall find it convenient to designate deviations from the mean by $x$ (instead of $x'$). If so, it will be stated that the origin of $x$ is at the mean or centroid.

**7. Moments in Standard Units.** The moments in standard units are denoted by the Greek letter alpha, $\alpha$. Thus for the $r$th moment in standard units, we have $\alpha_r = \frac{1}{N}\sum f_i t_i^r$. However, it is not necessary to transform the variates into $t$ units in order to compute the $\alpha$'s. We shall show that they are functions of the $\mu$'s. Thus

$$\alpha_r = \frac{1}{N}\sum f_i t_i^r \qquad \text{by definition}$$

$$= \frac{1}{N}\sum f_i \left(\frac{x_i - \bar{x}}{\sigma_x}\right)^r \qquad \text{from (9)}$$

$$= \frac{1}{\sigma_x{}^r}\,\frac{1}{N}\sum f_i (x_i - \bar{x})^r. \qquad \text{Why?}$$

Hence

$$\alpha_r = \frac{\mu_{r:x}}{(\sigma_x)^r} \qquad \text{Why?} \tag{10}$$

$$= \frac{\mu_{r:x}}{(\mu_{2:x})^{r/2}} \qquad \text{from (8).}$$

Letting $r = 1, 2, 3, 4$ in (10) we have

$$\begin{cases} \alpha_1 = \dfrac{\mu_{1:x}}{\sigma_x} = 0 \\ \alpha_2 = \dfrac{\mu_{2:x}}{\sigma_x{}^2} = 1 \\ \alpha_3 = \dfrac{\mu_{3:x}}{(\sigma_x)^3} \\ \alpha_4 = \dfrac{\mu_{4:x}}{(\sigma_x)^4}. \end{cases} \tag{10a}$$

It is obvious that $\alpha_1$ and $\alpha_2$ are abstract numbers. This is also the case for the other $\alpha$'s. In the expressions for $\alpha_3$ and $\alpha_4$ both numera-

tor and denominator are of the same dimension. That is to say, in $\alpha_3 = \mu_3/\sigma^3$ both numerator and denominator are the cubes of whatever unit is used in the original measurements, and therefore their ratio is of zero dimension, a pure number. Similarly, in $\alpha_4 = \mu_4/\sigma^4$ both numerator and denominator are the four powers of the same unit, and therefore $\alpha_4$ is an abstract number.

Some writers use $g_1$ instead of $\alpha_3$ and $g_2$ for $\alpha_4 - 3$.

**8. Use of $\alpha_3$ and $\alpha_4$.** Since $\alpha_1$ and $\alpha_2$ have the same values for all frequency distributions, their computation contributes nothing to the description or characterization of a distribution. But the values of $\alpha_3$ and $\alpha_4$ depend upon the shape of the histogram representing a distribution, and are therefore useful in distinguishing between types of distributions. Thus, we observe that

$$\mu_3 = \frac{1}{N}\sum f(x - \bar{x})^3$$

is a measure of asymmetry about the mean. If the variates are distributed symmetrically about $\bar{x}$ then $\mu_3 = 0$. But if the positive deviations from the mean outweigh the negative deviations then $\mu_3 > 0$, whereas if the negative deviations predominate, then $\mu_3 < 0$. Cubing the deviations gives a measure which is sensitive both to their size and sign but the result is in cubic units. Now symmetry, or lack of it, is not a function of the original units of measurement, so if we divide $\mu_3$ by $\sigma^3$ we get a pure number. Thus $\alpha_3$ is a satisfactory measure for comparing symmetry in distributions of different units of measurement.

The quantity $\alpha_4$ measures a characteristic called "kurtosis." It refers to the relative number of variates in the vicinity of the mean. More will be said about $\alpha_3$ and $\alpha_4$ later on. At this time, emphasis should be placed upon their calculation rather than upon the information which they yield.

Inasmuch as the $\alpha$'s are independent of the unit of measurement, they may be computed from the moments in the $u$ unit. Changing these moments into the $x$ unit would only introduce the same factor into the numerator and denominator, which would of course divide out. Thus:

$$\alpha_3 = \frac{\mu_{3:x}}{\sigma_x^3} = \frac{c^3\mu_{3:u}}{c^3\sigma_u^3} = \frac{\mu_{3:u}}{\sigma_u^3},$$

$$\alpha_4 = \frac{\mu_{4:x}}{\sigma_x^4} = \frac{c^4\mu_{4:x}}{c^4\sigma_u^4} = \frac{\mu_{4:u}}{\sigma_u^4}.$$

For Table 18 we have

$$\alpha_3 = \frac{-1.320}{(1.72)(1.31)} = -0.586,$$

$$\alpha_4 = \frac{9.4096}{(1.72)^2} = 3.18.$$

Although no limits can be placed on the possible values which $\alpha_3$ and $\alpha_4$ may take, it may be said that for the more common distributions $\alpha_4$ fluctuates around 3 and $\alpha_3$ is usually not more than 2 nor less than $-2$. We cannot go into the theoretical reasons for these values and we mention them here merely to guide the student as to what is a reasonable result to expect in the exercises in this book. In this connection, the inequality [1]

$$\alpha_4 \geqq \alpha_3^2 + 1$$

may also prove useful. When the numerical value of $\alpha_3$ is large, the distribution may be of the J-shaped type which is an extreme form of the asymmetrical type. However, these types cannot always be distinguished by elementary methods if the original data are not available.

**9. Summary.** The quantities $\bar{x}$, $\sigma_x$, $\alpha_3$, and $\alpha_4$ are called the descriptive constants of the distribution. They (together with $N$) are the "relatively few quantities" (§1) which, in certain cases, contain all the relevant information in the distribution. Table 20 will serve as a model for the procedure which the student should follow in computing these quantities. Of course, if the work is done on a computing machine, only the totals of the power sums need be recorded. The detail of the columns may be omitted. In Table 20, $c = 1$, so $\sigma_x = \sigma_u$. Obviously, this would not be true in general.

The calculation of the $\nu$'s proceeds naturally as an extension of the work required to compute $\bar{x}$ for a frequency distribution. Thus to obtain $\bar{x}$ we first compute $\nu_{1:u}$ and then obtain $\bar{x}$ from the relation

$$\bar{x} = c\bar{u} + x_0.$$

To obtain the standard deviation we need the value of $\nu_2$ because $\sigma_x$ is found from the relations

$$\mu_2 = \nu_2 - \bar{u}^2$$

$$\sigma_u = \sqrt{\mu_{2:u}}$$

$$\sigma_x = c\sigma_u.$$

[1] "A Note on Skewness and Kurtosis" — J. Ernest Wilkins, Jr. *Annals of Mathematical Statistics*, vol. 15 (1944), pp. 333-335.

The next chapter is devoted to a discussion of dispersion of which $\sigma_x$ is a measure. To be sure, the standard deviation is only one of several measures of dispersion, just as the mean is only one of several averages. But both the mean and the standard deviation play important roles in the theory and practice of statistics. It is important to master the pattern by which they are computed in a frequency distribution.

In order to compute $\alpha_3$ and $\alpha_4$ we first require $\nu_3$ and $\nu_4$ (in addition to $\nu_1$ and $\nu_2$). Then $\mu_3$ and $\mu_4$ are obtained from (5) and (6). Finally,

$$\alpha_r = \frac{\mu_{r:u}}{(\sigma_u)^r}$$

is computed for $r = 3$ and $r = 4$. The characteristics of a distribution which $\alpha_3$ and $\alpha_4$ describe will be discussed in Chapter VI and again in Part II. In elementary work they are less important than $\bar{x}$ and $\sigma_x$.

With regard to the number of decimal places to be retained in computations, the author agrees with Dr. Shewhart who says: "It does not appear feasible . . . to lay down simple, practical, and infallible rules." Reasons in support of this opinion are stated in his book,[1] pp. 79–80. For other remarks in this connection, the reader is referred to the books by Walker and Scarborough which are cited in our Introduction.

## Exercises

1. (*a*) What is the numerical value of the mean of any distribution of variates expressed in $t$ units?
   (*b*) What is the standard deviation of such a distribution? *Hint:* $\sigma_t = \sqrt{\alpha_2}$.
2. (*a*) Show that $(x - \bar{x}) = c(u - \bar{u})$ and hence that $t = (u - \bar{u})/\sigma_u$.
   (*b*) Show that we obtain the same results for the $\alpha$'s if we take

$$t = \frac{u - \bar{u}}{\sigma_u}.$$

3. Prove: If any constant is added algebraically to each variate of a series the values of $\mu_r$ for the new series will be identical with the corresponding values of $\mu_r$ of the original series.
4. Suppose each variate is multiplied by a constant. What effect would this have on $\bar{x}$, $\sigma_x$, $\alpha_3$, and $\alpha_4$?

[1] See footnote, p. 52.

**5.** Show that the standard deviation of $x$ may be written

$$\sigma_x = \left[\frac{1}{N}\sum f_i(x_i - \bar{x})^2\right]^{1/2}$$
$$= \left[\frac{1}{N}\sum f_i x_i^2 - \bar{x}^2\right]^{1/2}.$$

**6.** Prove the general relation

$$\mu_{r:x} = c^r \mu_{r:u}$$

of which the relations given in (7) are special cases when $r = 2, 3, 4$. *Hint:* $(x - \bar{x}) = c(u - \bar{u})$.

**7.** (*a*) Show that $\alpha_0 = 1$.
(*b*) Show that $\sigma^r = (\mu_2)^{r/2}$ in both the $x$ and $u$ units.

**8.** Prove from (4) that $\mu_2$ is less than or at most equal to $\nu_2$, the same unit being used in each case.

**9.** Find $\bar{x}$, $\sigma_x$, $\alpha_3$, and $\alpha_4$ for Iowa City rainfall using your results from Problem 4 of the preceding set of Exercises.
*Ans.*

$\bar{x} = 2.80$ in. $\qquad \alpha_3 = 1.29,$
$\sigma_x = 2.01$ in. $\qquad \alpha_4 = 4.58.$

**10.** Using Table 20 as a model find $\bar{x}$, $\sigma_x$, $\alpha_3$, and $\alpha_4$ for the distributions in §11, Chapter I, according to the direction of the instructor.

TABLE 20 — SPECIMEN WORKSHEETS FOR COMPUTING THE CHARACTERIZING CONSTANTS OF A DISTRIBUTION

*Subject: Span among Adult Males* (Table 13)

| $x$ | $f$ | $u$ | $uf$ | $u^2f$ | $u^3f$ | $u^4f$ | $(u+1)^4f$ |
|---|---|---|---|---|---|---|---|
| 58.5 | 1 | −11 | − 11 | 121 | −1,331 | 14,641 | 10,000 |
| 59.5 | 2 | −10 | − 20 | 200 | −2,000 | 20,000 | 13,122 |
| 60.5 | 1 | − 9 | − 9 | 81 | − 729 | 6,561 | 4,096 |
| 61.5 | 6 | − 8 | − 48 | 384 | −3,072 | 24,576 | 14,406 |
| 62.5 | 7 | − 7 | − 49 | 343 | −2,401 | 16,807 | 9,072 |
| 63.5 | 22 | − 6 | −132 | 792 | −4,752 | 28,512 | 13,750 |
| 64 5 | 55 | − 5 | −275 | 1,375 | −6,875 | 34,375 | 14,080 |
| 65.5 | 111 | − 4 | −444 | 1,776 | −7,104 | 28,416 | 8,991 |
| 66.5 | 146 | − 3 | −438 | 1,341 | −3,942 | 11,826 | 2,336 |
| 67.5 | 182 | − 2 | −364 | 728 | −1,456 | 2,912 | 182 |
| 68.5 | 229 | − 1 | −229 | 229 | − 229 | 229 | 0 |
| 69.5 | 265 | 0 | 0 | 0 | 0 | 0 | 265 |
| 70.5 | 263 | 1 | 263 | 263 | 263 | 263 | 4,208 |
| 71.5 | 217 | 2 | 434 | 868 | 1,736 | 3,472 | 17,577 |
| 72.5 | 176 | 3 | 528 | 1,584 | 4,752 | 14,256 | 45,056 |
| 73.5 | 132 | 4 | 528 | 2,112 | 8,448 | 33,792 | 82,500 |
| 74.5 | 82 | 5 | 410 | 2,050 | 10,250 | 51,250 | 106,272 |
| 75.5 | 48 | 6 | 288 | 1,728 | 10,368 | 62,208 | 115,248 |
| 76.5 | 20 | 7 | 140 | 980 | 6,860 | 48,020 | 81,920 |
| 77.5 | 16 | 8 | 128 | 1,024 | 8,192 | 65,536 | 104,976 |
| 78.5 | 12 | 9 | 108 | 972 | 8,748 | 78,732 | 120,000 |
| 79.5 | 3 | 10 | 30 | 300 | 3,000 | 30,000 | 43,923 |
| 80.5 | 1 | 11 | 11 | 121 | 1,331 | 14,641 | 20,736 |
| 81.5 | 2 | 12 | 24 | 288 | 3,456 | 41,472 | 57,122 |
| 82.5 | 1 | 13 | 13 | 169 | 2,197 | 28,561 | 38,416 |
| Sums | 2,000 | | 886 | 19,802 | 35,710 | 661,058 | 928,254 |
| (Sums)/$N$ | | | .443 | 9.901 | 17 855 | 330.529 | |
| | | | $\bar{u}$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | |

*Charlier's check:*

$$\sum(u+1)^4f = \sum u^4f + 4\sum u^3f + 6\sum u^2f + 4\sum uf + \sum f$$

$$928{,}254 = 661{,}058 + 4(35{,}710) + 6(19{,}802) + 4(886) + 2{,}000 = 928{,}254$$

*Computations:*

$\bar{x} = c\bar{u} + x_0 = (1)(.443) + 69.5 = 69.943$ in.

$\bar{u}^2 = .196249$

$\bar{u}^3 = .086938$, $\bar{u}^4 = .038514$

$\mu_2 = \nu_2 - \bar{u}^2$

$= 9.901 - .196249 = 9.704751$

$\sigma_u = \sqrt{9.704751} = 3.115$

$\sigma_x = c\sigma_u = (1)(3.115) = 3.115$ in.

$\mu_3 = \nu_3 - 3\nu_2\bar{u} + 2\bar{u}^3$

$= 17.855 - 3(9.901)(.443) + 2(.086938)$

$= 17.855 - 13.158429 + .173876$

$= 4.870447$

$\mu_4 = \nu_4 - 4\nu_3\bar{u} + 6\nu_2\bar{u}^2 - 3\bar{u}^4$

$= (330.529) - 4(17.855)(.443) + 6(9.901)(.96249) - 3(.028514)$

$= 330.529 - 31.639060 + 11.658368 - .115542$

$= 310.432769$

$\sigma_u^3 = (3.115)(9.704751) = 30.230299$

$\sigma_u^4 = (9.704751)^2 = 94.182192$

$$\alpha_3 = \frac{4.870447}{30.203299} = .161$$

$$\alpha_4 = \frac{310.432766}{94.082192} = 3.296$$

*Summary:*

$\bar{x} = 69.943$ in.; $\alpha_3 = 0.161$;

$\sigma_x = 3.115$ in.; $\alpha_4 = 3.296$.

**10. Sheppard's Corrections.** The moments of a frequency distribution are computed on the assumption that each variate value in a class interval has the value of the class mark for that interval. This has the effect of replacing the actual data by somewhat fictitious data assigned arbitrarily at the central values of the intervals. Evidently a very coarse grouping might be misleading and it can be shown mathematically that the above assumption introduces a systematic error, called a grouping error, in the results obtained for the second and fourth moments about the mean but does not affect $\mu_1$ and $\mu_3$. To eliminate this systematic tendency certain corrections are applied to $\mu_2$ and $\mu_4$.

The derivation of these corrections is beyond the scope of an elementary course, but it may be worth while to see why it is that corrections are necessary for some moments and not for others. The following argument is intended only as a pedagogical device to give a plausible explanation. Suppose a smooth curve represents the

true frequency distribution while the histogram represents the distribution with class marks as the variates. Since the moments are computed from the distribution represented by the histogram, we scarcely expect our results to be exactly the values of the moments of the true distribution, which are, of course, what we seek. In using the distribution represented by the histogram, we are neglecting, for each rectangle, the little area under the curve shaded $A$ and substituting for it the little area shaded $B$. Suppose that, in general, $B$ is a little larger than $A$, as shown in Figure 12. The excess of $B$

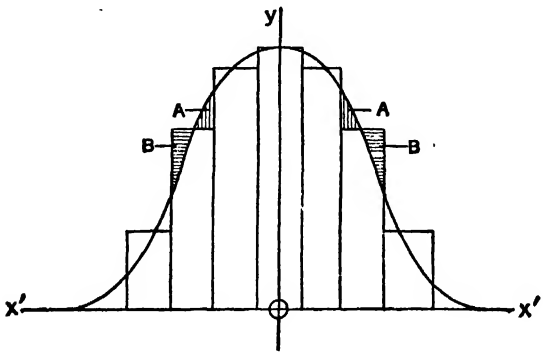



FIG. 12

over $A$ for those rectangles to the left of $\bar{x}$ will be negative; the corresponding excess for those rectangles to the right of $\bar{x}$ will be positive. This may be readily understood by considering these little areas as approximate triangles whose bases are negative or positive according as they are to the left or right of $\bar{x}$. These excesses for all the rectangles, both positive and negative, are involved in taking the summation $\sum f_i(x_i - \bar{x})^r$ for the moments. When $r$ is an odd number, as 1 or 3, the excesses show up with their algebraic signs and therefore, over the range of the distribution, the positive excesses just about offset the negatives ones. But in the case of the even moments, all the excesses now become positive so that the errors accumulate and the final results for these moments are too large.

To reduce these errors due to grouping, W. F. Sheppard has demonstrated[1] that the following corrections should be applied. It should

[1] Students familiar with more advanced mathematics will find an interesting discussion of systematic errors and references to papers dealing with Sheppard's corrections in an article by H. C. Carver, *Annals of Mathematical Statistics*, vol. 7, p. 154.

be noticed that as we state them here they should be applied only where the class interval is unity, *i.e.*, in the $u$ unit.

$$\text{Corrected } \mu_{2:u} = \text{uncorrected } \mu_{2:u} - \frac{1}{12}$$

$$\text{Corrected } \mu_{3:u} = \text{uncorrected } \mu_{3:u}$$

$$\text{Corrected } \mu_{4:u} = \text{uncorrected } \mu_{4:u} - \frac{1}{2}(\text{uncorrected } \mu_{2:u}) + \frac{7}{240}$$

$$\left(\frac{1}{12} = 0.08333, \frac{7}{240} = 0.02917\right).$$

*Example.* For Table 18 we have

$$\begin{aligned}
\text{Corrected } \mu_{2:u} &= 1.720 - 0.083 = 1.637 \\
\sigma_u &= \sqrt{1.637} = 1.28 \\
\text{Corrected } \sigma_x &= 10(1.28) = 12.8\% \\
\text{Corrected } \mu_{4:u} &= 9.4096 - (1.72)/2 + 7/240 \\
&= 8.5788 \\
\therefore\ \alpha_4 &= 8.5788/(1.637)^2 = 3.20
\end{aligned}$$

The values of $\bar{x}$ and $\mu_3$ remain unchanged.

Sheppard's corrections are valid only for the bell-shaped types of distributions. They are not applicable to the J-shaped or U-shaped types. Moreover, they constitute a refinement which may not always be consistent with the degree of accuracy in the original data. The errors of grouping (not mistakes) are usually small compared with the errors existing in the raw data. So, it seems that little would be gained by their use in a first course. We will occasionally use them in an illustration.

# CHAPTER V

## MEASURES OF DISPERSION

**1. Introduction.** The concept of variability is fundamental today not only in the social sciences but also in the so-called exact physical sciences. Modern scientific method recognizes the existence of physical, moral, and mental inequalities. The principle of variability has come to be accepted as the natural order in social, economic, and physical phenomena. This principle is the very essence of the statistical nature of mass phenomena. In this connection, R. A. Fisher says:[1]

> The conception of statistics as the study of variation is the natural outcome of viewing the subject as the study of populations; for a population of individuals in all respects identical is completely described by a description of any one individual, together with the number in the group. The populations which are the object of statistical study always display variation in one or more respects. To speak of statistics as the study of variation also serves to emphasize the contrast between the aims of modern statisticians and those of their predecessors. For, until comparatively recent times, the vast majority of workers in this field appear to have had no other aim than to ascertain aggregate, or average, values. The variation itself was not an object of study, but was recognized rather as a troublesome circumstance which detracted from the value of the average. . . . Yet, from the modern point of view, the study of the causes of variation of any variable phenomena, from the yield of wheat to the intellect of man, should be begun by the examination of the variation which presents itself. The study of variation leads immediately to the concept of a frequency distribution.

It is clearly important, therefore, in studying a distribution, to describe how the variates are clustered or scattered around an average. Figure 13 shows how two distributions may even have the same mean and total frequency, yet differ considerably in variation from the mean. Such variation is commonly called dispersion, variability, or spread.

We will consider three measures of dispersion: *Quartile Deviation*,

[1] R. A. Fisher, *Statistical Methods for Research Workers*, p. 3. Oliver and Boyd, London.

*Mean Deviation*, and *Standard Deviation*, of which the last is by far the most important.

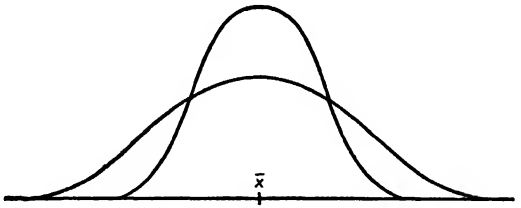



FIG. 13. TWO DISTRIBUTIONS DIFFERING IN DISPERSION

**2. The Quartile Deviation.** Just as the median selects one point of division, we may now take two additional points such that they, together with the median, divide the whole distribution into four equal parts. These points are called the quartile values.

The first quartile, denoted by $Q_1$, is that value of $x$ for which $cum\ f = N/4$. That is, one-fourth of all the variates in the distribution are smaller in value than $Q_1$ and three-fourths of them are larger than $Q_1$. The second quartile $Q_2$ is that value of $x$ for which $cum\ f$ is $N/2$ and is therefore the median. The third quartile, denoted by $Q_3$, is that value of $x$ for which $cum\ f = 3N/4$. Hence fifty per cent of the total frequency is included between $Q_1$ and $Q_3$.

Half of the distance between $Q_3$ and $Q_1$ is called the *semi-interquartile range* or *quartile deviation* and will be denoted by $Q$. Thus,

$$(1) \qquad Q = \frac{Q_3 - Q_1}{2}$$

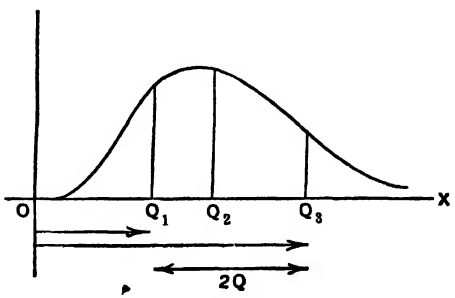



FIG. 14

It should be noted that the median does not necessarily come at the mid-point of $2Q$, *i.e.*, that a distance $Q$ laid off on either side of $Q_2$ would not necessarily reach to $Q_1$ and $Q_3$. (See Figure 14.) (For a symmetrical distribution, to be considered later, this would be true.)

As a measure of dispersion, $Q$ gives a fairly good idea of the spread of the variates, and is suitable as such a measure in those cases where the median would be used as an average. The quartile values $Q_1$

and $Q_3$ are found, like the median, by interpolation in the cumulative frequency table.

*Example.* (*a*) Find the median and the quartile deviation for the distribution of IQ's in Table 6 (§10, Chapter I). (*b*) Illustrate the measures found in (*a*) by means of a *cum f* graph.

| *End-x* | *Cum f* |
|---|---|
| 54 5 | 0 |
| 64.5 | 3 |
| 74.5 | 24 |
| 84.5 | 102 |
| ← $Q_1$ | |
| 94.5 | 284 |
| ← Med. | |
| 104.5 | 589 |
| ← $Q_3$ | |
| 114.5 | 798 |
| 124.5 | 879 |
| 134.5 | 900 |
| 144 5 | $N$ = 905 |

*Solution:*

$$N/4 = 226.25, \quad N/2 = 452.5, \quad 3N/4 = 678.75$$

$$\frac{Q_1 - 84\,5}{10} = \frac{226.25 - 102}{284 - 102}, \qquad Q_1 = 91.3$$

$$\frac{Q_2 - 94.5}{10} = \frac{452.5 - 284}{589 - 284}, \qquad Q_2 = 100.02$$

$$\frac{Q_3 - 104.5}{10} = \frac{678.75 - 589}{798 - 589}, \qquad Q_3 = 108.8$$

$$Q = \frac{Q_3 - Q_1}{2} = 8.75.$$

Figure 15 explains graphically the measures obtained by interpolation from a *cum f* table. For convenience in drawing the figure, the quartile labels are put on vertical lines. But one should remember that the quartiles are values of $x$ and that it is the horizontal distances of the lines from the $y$-axis that represent these measures.

## Exercises

'1. Criticize the following "definitions":

$$Q_1 = \frac{N}{4}, \quad Q_2 \quad \frac{N}{2}, \quad Q_3 = \frac{3N}{4}.$$

2. Find $Q_1$ and $Q_3$ from the cumulative frequency table which you made to obtain the median for the Glasgow schoolgirl distribution. (Exercise 5 on page 52.)
3. Find the quartile deviation $Q$ from your results in Exercise 2.
4. Find $Q_1$, $Q_2$, $Q_3$ for the distribution in Table 12, and compute $Q$.
5. Compute the value of the semi-interquartile range for other distributions at the direction of the instructor.

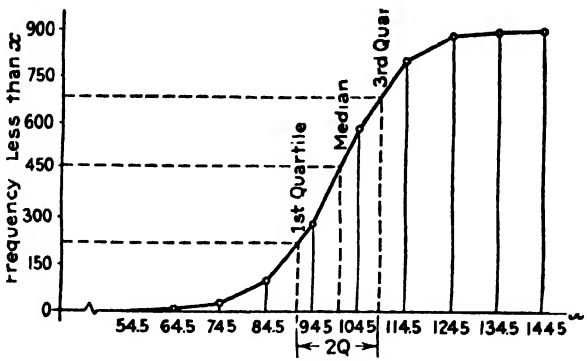



FIG. 15

6. The $m$th *percentile* $P_m$ of a frequency distribution is that value of the variable $x$ for which cum $f = mN/100$, where $m = 1, 2, \cdots, 99$. The 10th, 20th, 30th, $\cdots$, percentiles are called *deciles*. Therefore, the $n$th decile $D_n$ is that value of $x$ for which cum $f = nN/10$, where $n = 1, 2, \cdots, 9$. Compute several percentiles and deciles of a distribution in the text.

**3. Mean Deviation.** As a measure of variation about a central value, it would seem appropriate to take an average of all the deviations about that central value. In the mean deviation (MD) about the mean this is precisely what we do, namely, we find the arithmetic mean of the numerical values of the deviations about the mean. In summing the deviations, their absolute values are used because regardless of whether deviations are positive or negative they have the same influence on the amount of variation. Moreover, if their algebraic signs are taken account of, the sum of such deviations is

zero (Theorem VI of Chapter III). Hence we sum them treating all deviations as positive.

In mathematical symbols, vertical bars denote absolute values, so we have[1]

$$\textbf{MD} = \frac{1}{N}\sum f_i \left| x_i - \bar{x} \right|, \tag{2}$$

if the $x$ unit is used. When the class interval is the unit, we have

$$\textbf{MD} = \frac{1}{N}\sum f_i \left| u_i - \bar{u} \right| \tag{3}$$

and

$$\textbf{MD}\ (x\ unit) = c \times \textbf{MD}\ (u\ unit). \tag{4}$$

It can be proved that the essentially positive function

$$y = \frac{1}{N}\sum f_i (x_i - A)^2$$

is a minimum when $A = \bar{x}$. (See Theorem II, page 99. Also by the calculus $dy/dA = 0$ when $A = \bar{x}$.) It was in a similar investigation to find the value of $B$ for which the function

$$y = \frac{1}{N}\sum f_i \left| x_i - B \right|$$

is a minimum, that the median was discovered. When $B$ is the median this function is a minimum.[2] This property of the median has some statistical importance in connection with the geographical location of centers of industry and population.[3] Custom has established the use of the mean rather than the median in this measure. Hence "*mean deviation*" usually refers to the mean deviation from the mean. It is also called "average deviation."

[1] Since all the data are not concentrated at the midpoints of the intervals, a grouping error is involved here as in the formula for $\sigma$ (§10, Chapter IV). But the mean deviation is used so infrequently that discussion here of the appropriate correction hardly seems warranted. Those who may be interested will find a more precise formula in the *Handbook of Mathematical Statistics* — Rietz and others.

[2] For a proof see reference 16, our Introduction.

[3] See p. 85 of *Elements of Statistics* — Davis and Nelson. Principia Press.

*Example.* Find the mean deviation for the grades in Table 18 where the mean value of $x$ is 72.5.

| $x$ | $f$ | $\lvert x - \bar{x}\rvert$ | $f\lvert x - \bar{x}\rvert$ |
|---|---|---|---|
| 34.5 | 2 | 38 | 76 |
| 44.5 | 3 | 28 | 84 |
| 54.5 | 11 | 18 | 198 |
| 64.5 | 20 | 8 | 160 |
| 74.5 | 32 | 2 | 64 |
| 84.5 | 25 | 12 | 300 |
| 94.5 | 7 | 22 | 154 |
| Total | 100 | | 1036 |

$$\text{MD} = \frac{1036}{100} = 10.36.$$

What was $\sigma$ for this distribution?

The absolute value of a variable $x'$, denoted by the symbol $|x'|$, is not very tractable in mathematical operations. Therefore the mean deviation is not favored by mathematicians since it is unwieldy in the more theoretical and mathematical discussions. Its chief use is in experimental work where occasional large and erratic deviations are likely to occur. In such cases the standard deviation would tend to emphasize these deviations.

If $m$ of the $N$ variates are greater than the mean, $\bar{x}$, then the mean deviation may be written

$$\text{MD} = \frac{1}{N}\{(\text{sum of variates greater than } \bar{x}) - m\bar{x}$$

$$= \frac{1}{N}\left\{\sum_{x_i > \bar{x}} f_i x_i - \bar{x}\sum_{x_i > \bar{x}} f_i\right\}.$$

The student is given a hint in Exercise 34 at the end of Part I on how to prove a similar formula for $x_i < \bar{x}$.

**4. The Standard Deviation.** To overcome the difficulty of negative deviations and the use of absolute value signs, the deviations about the mean may be squared and the mean of these squares taken.

To get back into the original linear units, we take the positive square root of this result, and have

$$\sigma_x = \left[\frac{1}{N}\sum f_i(x_i - \bar{x})^2\right]^{1/2} \tag{5}$$

as defined before. The standard deviation measures the same kind of phenomenon as the mean deviation and this approach to it is frequently satisfactory to a student who otherwise finds it difficult to understand.[1]

For a common type of distribution, the standard deviation is approximately twenty-five per cent greater than the mean deviation. Speaking more accurately, this is true of a normal distribution (to be considered in Chapter VI) for which the relation is MD = $\frac{4}{5}\sigma$ (approximately).

It is often convenient to have a name for " the square of the standard deviation," and for this purpose the term " variance " has been introduced. Thus $\sigma$ denotes standard deviation and $\sigma^2$ denotes variance.

Although definition (5) is the basic concept which the student should have for the standard deviation, nevertheless in actual practice it is seldom desirable to compute $\sigma$ directly from that definition. For a frequency distribution the method is shown in the chapter on moments. However, we will give an additional illustration here.

*Example.* Find the mean and the standard deviation of Table 9, using Charlier's check and Sheppard's correction.

*Solution:* (See Table 21, p. 88.)

Charlier's check: $\sum f(u+1)^2 = \sum fu^2 + 2\sum fu + N$

$$2471 = 2365 + 2(-447) + 1000 = 2471$$

*Computations:*

$$\bar{x} = 49.5 + 4(-.447) = 47.712 \text{ lbs.}$$

$$\mu_{2:u} = \nu_2 - (\bar{u}^2) = 2.165$$

[1] The term " standard deviation " was proposed by Pearson and is now used by almost all English writers. As originally defined by Pearson, this is the square root of the mean of the squares of deviations *taken from the mean of the distribution*, and is not to be used when deviations are measured from any other reference point. Pearson uses the term " root-mean-square " for a similar measure when the deviations are taken around any origin other than the mean. — Walker, *History of Statistical Method*, p. 54.

Using Sheppard's corrections,

$$\text{Corrected } \mu_2 = 2.165 - .083 = 2.082$$
$$\mu_{2:x} = 16(2.082) = 33.312$$
$$\sigma_x = \sqrt{33.312} = 5.772 \text{ lbs.}$$

TABLE 21 — WEIGHTS OF GLASGOW SCHOOL CHILDREN

| *Weight* ($x$) | $f$ | $u$ | $fu$ | $fu^2$ | $f(u+1)^2$ |
|---|---|---|---|---|---|
| 29.5 lbs. | 1 | −5 | − 5 | 25 | 16 |
| 33.5 | 14 | −4 | − 56 | 224 | 126 |
| 37.5 | 56 | −3 | −168 | 504 | 224 |
| 41.5 | 172 | −2 | −344 | 688 | 172 |
| 45.5 | 245 | −1 | −245 | 245 | 0 |
| 49.5 | 263 | 0 | 0 | 0 | 263 |
| 53.5 | 156 | 1 | 156 | 156 | 624 |
| 57.5 | 67 | 2 | 134 | 268 | 603 |
| 61.5 | 23 | 3 | 69 | 207 | 368 |
| 65.5 | 3 | 4 | 12 | 48 | 75 |
| Sums | 1000 | | −447 | 2365 | 2471 |
| (Sums)/$N$ | 1 | | −.447 $\bar{u}$ | 2.365 $\nu_2$ | |

It will be proved later that for a certain ideal type of distribution which is often approximated in practical statistics the range $\bar{x} \pm \sigma_x$ includes about two thirds of the variates. Assuming the above distribution is of this type we could say that about two thirds of the children weighed between 42 pounds and 53.5 pounds. Such a statement assists one in comprehending certain characteristics of the data though the distribution actually may not be before him.

It is understood that the method of computation described above is to be used when the class marks are equispaced. If the class intervals are unequal we must choose $c = 1$ unless the $x$'s denoting the class marks have a common factor $c$. When $c = 1$, $u$ becomes $u = x - x_0$, and the work may be simplified a little by an appropriate choice of $x_0$.

## Exercises

1. (*Pearson*). The following data represent the percentage of ash-content in 280 wagon tests of a certain kind of coal. Find the mean and the standard deviation of the distribution:

| *Percentage Ash-Content* | *Frequency* |
|---|---|
| 3.0– 3.9 | 1 |
| 4.0– 4.9 | 7 |
| 5.0– 5.9 | 28 |
| 6.0– 6.9 | 78 |
| 7.0– 7.9 | 84 |
| 8.0– 8.9 | 45 |
| 9.0– 9.9 | 28 |
| 10.0–10.9 | 7 |
| 11.0–11.9 | 2 |

*Ans.* $\bar{x} = 7.35\%$, $\sigma_x = 1.36\%$.

2. (*Camp*). Find the mean wage and the standard deviation of the following data:

| *Class* | *Frequency* |
|---|---|
| \$4.50– 5.99 | 43 |
| 6.00– 7.49 | 99 |
| 7.50– 8.99 | 152 |
| 9.00–10.49 | 178 |
| 10.50–11.99 | 160 |
| 12.00–13.49 | 40 |
| 13.50–14.99 | 25 |
| 15.00–16.49 | 3 |

*Ans.* $N = 700$, $\bar{x} = \$9.42$, $\sigma_x = \$2.19$.

3. Given $\sigma_x = 2.19$ for the following $(x, f)$ distribution, find $\sigma_v$ and $\sigma_u$ for the $(v, f)$ and $(u, f)$ distributions, respectively.

| $f$ | 43 | 99 | 152 | 178 | 160 | 40 | 25 | 3 |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1.5 | 3.0 | 4.5 | 6.0 | 7.5 | 9.0 | 10.5 |
| $v$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $u$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |

What relation and theorem in Chapter IV does this illustrate?

4. Find the variance $\sigma_x^2$ of Table 16 (§8, Chapter III).
5. Compute the value of the ratio $MD/\sigma$ for the data in Exercise 1 above.
6. Find the mean and standard deviation for the data in Table 10.

7. Find the mean and standard deviation for the data in Table 11.
8. Transform the variates of the following distribution into standard units:

| $x$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f$ | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |
| $t$ | Some answers: | | | | | 1/3 | 1 | 5/3 | 7/3 | 3 |

**5. Relative Dispersions.** The full significance of different values of $\sigma$ can be obtained only by experience, but it is obvious that a small standard deviation indicates that the variates are closely clustered about the mean; whereas a large standard deviation indicates that these values are spread out widely from the mean. (See Figure 13.)

The size of variates usually influences not only the mean but also deviations from the mean. In other words, the magnitudes of the deviations from the mean seem to be dependent, in some degree, upon the magnitude of the mean. In comparing dispersion in distributions, we may correct for differences in the average magnitudes of positive variates by taking the ratio of the standard deviation to the mean. Thus, the quantity

$$V = \frac{\sigma_x}{\bar{x}} \tag{6}$$

is known as the *coefficient of variation*. It is obviously an abstract number, being independent of the units of measurement, and it is usually expressed as a percentage.

The use of (6) may be misleading in situations where the origin from which the data are measured is somewhat arbitrary. Cases in point are temperature measurements and certain psychological data. Further discussion of such limitations of (6) will be found in references 2, 14, and 15, listed in the Introduction.

**6. Scaling a Distribution in Terms of $\sigma$.** Suppose we lay off intervals of length $\sigma$ on either side of the mean (Figure 16). Then for a certain type of distribution known as the normal curve (which will be considered in the next chapter) the following properties can be proved:

(1) The percentage of the total frequency lying outside the range $\bar{x} \pm \sigma$ is 32% approximately.

(2) The percentage outside $\bar{x} \pm 2\sigma$ is 5% approximately.

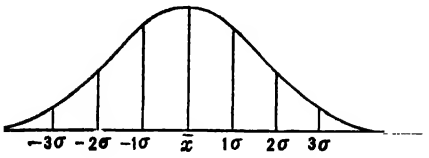



FIG. 16

(3) The range $\bar{x} \pm 3\sigma$ includes practically the whole distribution, *i.e.*, the total range is $6\sigma$ approximately.

The student will recognize that these ranges are, in standard units, $t = \pm 1$, $t = \pm 2$, $t = \pm 3$, respectively. These results follow from the relation

$$t = \quad , \quad x = \bar{x} + t\sigma.$$

Sometimes it is important in a statistical analysis to know how nearly the given variates are distributed in accordance with the

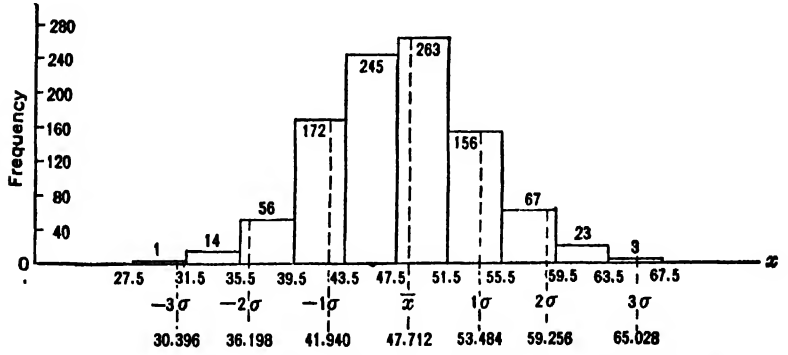



FIG. 17 — DISTRIBUTION OF TABLE 21 SCALED OFF IN UNITS OF $\sigma$

above property of the normal curve. The distribution of Table 21 has been scaled off in this manner, with the results shown in Table 22. Figure 17 will be helpful in verifying them.

We will verify here the 34.8% given in Table 22, and the student is asked to verify the others in Exercise 2. The range $\bar{x} \pm \sigma$ (Figure 17) evidently includes all the variates represented by the two central rectangles and proportionate parts of the two adjoining rectangles. From 39.50 to 41.94 is 2.44, and since the variates are assumed to be uniformly distributed over the class interval we have 172(2.44/4) =

104.92 for the proportionate number to be excluded in the class 39.5–43.5. Hence the number below $\bar{x} - \sigma$ is $(1 + 14 + 56 + 104.92) = 175.92$. Similarly, from 53.484 to 55.5 is 2.016, and we have $156(2.016/4) = 78.624$ as the proportionate number excluded in the class 51.5–55.5. Hence the total above $\bar{x} + \sigma$ is $(78.624 + 67 + 23 + 3) = 171.624$. So the total number outside $\bar{x} \pm \sigma$ is $(171.624 + 175.92) = 348$ or 34.8% of the 1000 variates. This result could also be obtained as follows: By forming a *cum f* table and interpolating in the *end x* column we find

TABLE 22 — RESULTS OF SCALING OFF TABLE 21

| $\bar{x} = 47.712$, $\sigma_x = 5.772$ | | *Range* | *Frequency outside the given range* | |
|---|---|---|---|---|
| | | | *Number* | *Percent* |
| $\bar{x} - \sigma = 41.940$ | $\bar{x} + \sigma = 53.484$ | $\bar{x} \pm \sigma$ | 348 | 34.8 |
| $\bar{x} - 2\sigma = 36.198$ | $\bar{x} + 2\sigma = 59.256$ | $\bar{x} \pm 2\sigma$ | 60 | 6.0 |
| $\bar{x} - 3\sigma = 30.396$ | $\bar{x} + 3\sigma = 65.028$ | $\bar{x} \pm 3\sigma$ | 3 | 0.3 |

| | |
|---|---|
| *cum f* at $x = 53.484$: | 828 |
| *cum f* at $x = 41.940$: | 176 |
| Number in the $(\bar{x} \pm \sigma_x)$ interval: | 652 |
| Number outside this interval: | 348 |

**7. Semi-interquartile Range in Terms of $\sigma$.** The range $(Q_3 - Q_1)/2$ when expressed in units of $\sigma$ has a significance in a normal distribution, as will be shown later. We will denote this by $s$; hence

$$s = \frac{Q_3 - Q_1}{2\sigma}, \quad \text{and} \quad s = \frac{Q}{\sigma}.$$

For the present we merely calculate its value in the exercises below.

## Exercises

1. Find the mean and standard deviation for the distribution of Lengths of Telephone Calls, given in Table 8 (Chapter I). Use Charlier's check.
2. In the three distributions named, show that the percentages outside $\bar{x} + t\sigma$ for $t = \pm 1$, $\pm 2$, and $\pm 3$, are as stated in Table 23. Verify also the values of $s$.

TABLE 23

| *Distribution* | *N* | *Percent Outside* | | | *s* |
|---|---|---|---|---|---|
| | | $\bar{x} \pm \sigma$ | $\bar{x} \pm 2\sigma$ | $\bar{x} \pm 3\sigma$ | |
| *Glasgow girls* | 1000 | 34.8 | 6.0 | 0.3 | 0.675 |
| *Telephone calls* | 995 | 32.7 | 5.0 | 0.4 | 0.69 |
| *Span* | 2000 | 31.8 | 4.2 | 0.5 | 0.665 |

**8. *N* Small. Ungrouped Data.** When $N$ is small it is seldom desirable to attempt an arrangement of the variates into a frequency distribution. Moreover, in this case, the values of $\alpha_3$ and $\alpha_4$ are not usually needed because the applications of these measures relate to characteristics of large distributions. Therefore, only the mean and standard deviation are usually required for a small set of ungrouped

TABLE 24 — AVERAGE YIELDS OF CORN IN BUSHELS PER ACRE FOR A CERTAIN SECTION IN ILLINOIS FROM 1901–1920

| *Year* | *Yield* ($x$) | $u$ | $u^2$ |
|---|---|---|---|
| 1901 | 21 | −15 | 225 |
| 1902 | 39 | 3 | 9 |
| 1903 | 32 | − 4 | 16 |
| 1904 | 37 | 1 | 1 |
| 1905 | 40 | 4 | 16 |
| 1906 | 36 | 0 | 0 |
| 1907 | 36 | 0 | 0 |
| 1908 | 32 | − 4 | 16 |
| 1909 | 36 | 0 | 0 |
| 1910 | 39 | 3 | 9 |
| 1911 | 33 | − 3 | 9 |
| 1912 | 40 | 4 | 16 |
| 1913 | 27 | − 9 | 81 |
| 1914 | 29 | − 7 | 49 |
| 1915 | 36 | 0 | 0 |
| 1916 | 30 | − 6 | 36 |
| 1917 | 38 | 2 | 4 |
| 1918 | 36 | 0 | 0 |
| 1919 | 36 | 0 | 0 |
| 1920 | 35 | − 1 | 1 |
| Totals | $N = 20$ | −32 | 488 |

data. The following methods will help the student become familiar with the several formulas for $\sigma$, which may be used in this case.

*Method I.* The indirect method involving the $u$ unit may still be used for finding the first and second moments. Since each variate is being treated separately $f = 1$, and we compute the values of $\nu_r = \frac{1}{N}\sum u^r$ for $r = 1$ and 2. If the values of $x$ are unequally spaced we take $c = 1$ and let $u = x - x_0$ which changes the origin but not the units. In other words, the procedure is the same as for a frequency distribution except that $f = 1$ and $c = 1$.

*Example.* Find the mean and standard deviation for Table 24. $N = 20$. We choose $x_0 = 36$.

TABLE 25

| $x$ | $x' = x - \bar{x}$ | $x'^2$ |
|---|---|---|
| 21 | −13.4 | 179.56 |
| 27 | − 7.4 | 54.76 |
| 29 | − 5.4 | 29.16 |
| 30 | − 4.4 | 19.36 |
| 32 | − 2.4 | 5.76 |
| 32 | − 2.4 | 5.76 |
| 33 | − 1.4 | 1.96 |
| 35 | 0.6 | .36 |
| 36 | 1.6 | 2.56 |
| 36 | 1.6 | 2.56 |
| 36 | 1.6 | 2.56 |
| 36 | 1.6 | 2.56 |
| 36 | 1.6 | 2.56 |
| 36 | 1.6 | 2.56 |
| 37 | 2.6 | 6.76 |
| 38 | 3.6 | 12.96 |
| 39 | 4.6 | 21.16 |
| 39 | 4.6 | 21.16 |
| 40 | 5.6 | 31.36 |
| 40 | 5.6 | 31.36 |
| 688 | $\sum \lvert x' \rvert = 73.6$ | 436.80 |

*Computations:*

$$\nu_1 = \bar{u} = -\frac{32}{20} = -1.6; \quad \bar{x} = x_0 + \bar{u} = 36 - 1.6$$
$$= 34.4 \text{ bushels}$$

$$\nu_2 = \frac{488}{20} = 24.40; \quad \mu_2 = \nu_2 - \bar{u}^2 = 21.84.$$

Therefore,

$$\sigma_x = \sigma_u = \sqrt{21.84} = 4.67 \text{ bushels.}$$

*Method II.* When $f = 1$, formula (5) becomes

$$\sigma_x = \left[\frac{1}{N}\sum(x_i - \bar{x})^2\right]^{1/2}, \tag{7}$$

and sometimes it is best to compute the standard deviation directly from this definition, without the use of the $u$ unit. Thus the origin is placed at the mean and all indirect methods are abandoned. If the mean deviation is also desired, clearly this method should be used. It is exemplified in Table 25 for the preceding example, and the variates have been arranged in order of magnitude.

$$\bar{x} = \frac{688}{20} = 34.4 \text{ bushels}$$

$$\sigma_x^2 = \frac{436.80}{20} = 21.84$$

$$\sigma_x = 4.67 \text{ bushels}$$

$$\text{MD} = \frac{1}{N}\sum|x'| = \frac{73.6}{20} = 3.68 \text{ bushels.}$$

*Method III.* From the relation

$$\mu_2 = \nu_2 - (\nu_1)^2$$

we have

$$\mu_2 = \frac{1}{N}\sum x^2 - \bar{x}^2$$

when $f = 1$. Therefore $\sigma$ may be written

$$\sigma_x = \left[\frac{1}{N}\sum x^2 - \bar{x}^2\right]^{1/2}. \tag{8}$$

TABLE 26

| $x$ | $x^2$ |
|---|---|
| 21 | 441 |
| 27 | 729 |
| 29 | 841 |
| 30 | 900 |
| 32 | 1024 |
| 32 | 1024 |
| 33 | 1089 |
| 35 | 1225 |
| 36 | 1296 |
| 36 | 1296 |
| 36 | 1296 |
| 36 | 1296 |
| 36 | 1296 |
| 36 | 1296 |
| 37 | 1369 |
| 38 | 1444 |
| 39 | 1521 |
| 39 | 1521 |
| 40 | 1600 |
| 40 | 1600 |
| 688 | 24,104 |

This method is perhaps the best when the values of $x$ are not large or when a table of squares is available. It is illustrated below for the preceding example. (See Table 26.)

*Computations:*

$$\bar{x} = \frac{1}{N}\sum x = \frac{688}{20} = 34.4 \text{ bushels}$$

$$\bar{x}^2 = (34.4)^2 = 1183.36$$

$$\frac{1}{N}\sum x^2 = \frac{24104}{20} = 1205.20$$

$$\sigma_x = [1205.20 - 1183.36]^{1/2}$$

$$= (21.84)^{1/2}$$

$$= 4.67 \text{ bushels.}$$

## Miscellaneous Exercises

1. (*a*) Verify that the algebraic sum of the numbers in the $x'$ column of Table 25 is zero.
   (*b*) Verify the value of mean deviation given for Table 25.

**2.** Using your own judgment as to the most appropriate method, find the mean and standard deviation for each of the two sets of data, $x_1$ and $x_2$:

| | | *Answers* |
|---|---|---|
| $x_1$ | 88 95 68 73 75 88 57 68 62 79 73 74 78<br>80 57 65 69 74 78 72 59 47 56 67 43 | $\bar{x}_1 = 69.80$<br>$\sigma_1 = 12.13$ |
| $x_2$ | 82 86 75 78 72 79 63 65 67 75 68 70 79<br>78 51 58 65 69 68 83 80 42 43 48 47 | $\bar{x}_2 = 67.64$<br>$\sigma_2 = 12.68$ |

**3.** Complete the computations and find the mean and variance of the following distribution:

| *Data* | | *Computations* | | |
|---|---|---|---|---|
| $y$ | $f$. | $v$ | $vf$ | $v^2f$ |
| 67.86 | 7 | −16.94 | | |
| 72.14 | 14 | −12.66 | | |
| 81.87 | 32 | − 2.93 | | |
| 84.80 ← $y_0$ | 49 | 0 | | |
| 85.73 | 55 | .93 | | |
| 90.92 | 54 | 6.12 | | |
| 95.57 | 35 | 10.77 | | |
| 105.00 | 14 | 20.20 | | |

*Hint.* Here we let $v = y - y_0$. Then $\bar{y} = \bar{v} + y_0$, and $\sigma_y^2 = \sigma_v^2$ since $c = 1$. (See Theorem on p. 69.)

*Ans.* $\bar{y} = 87.31$, $\sigma_y^2 = 56.66$.

**4.** Data have been gathered showing the points scored on a mental test by 290 prospective employees and the per cent of standard production attained by these same 290 persons after being employed.[1] The following statistics were obtained:

Mental test: mean = 43.33 pts.
$\sigma$ = 9.25 pts.
Productive ability: mean = 92.02%
$\sigma$ = 24.47%

(*a*) Compare the relative dispersion in mental test and productive ability.
(*b*) What factors, other than mental level, may have affected dispersion under factory conditions?

[1] Wembridge, "Experiment and Statistics in the Selection of Employees," *Journal of the American Statistical Association*, March 1923, p. 605.

5. Read and abstract the article "Variability," *Journal of Educational Research*, vol. 4, no. 3, pp. 221–26.
6. Find the median for Table 26.
7. Find $\bar{x}$, $\sigma_x$, MD, and $Q$ for the following distribution.

| *mid-x* | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| *f* | 1 | 4 | 6 | 4 | 1 |

8. Show that (8) may be written as follows:

$$\sigma_x = \frac{[N\sum x^2 - (\sum x)^2]^{1/2}}{N}.$$

9. If the variates are all equal, say each $x_i = k$, show that $\bar{x} = k$ and $\sigma = 0$.
10. For a set of ungrouped data it is found that $N = 15$, $\sum x = 480$, $\sum x^2 = 15{,}735$. Find $\bar{x}$ and $\sigma_x$.
11. Find the variance of the following data.

5.7 6.2 6.5 6.0 6.3 5.8 5.7 6.0 6.0 5.8

*Ans.* $\sigma_x^2 = .064$.

12. Prove the identity:

$$(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2 = (x_1^2 + x_2^2 + \cdots + x_N^2) - N\bar{x}^2.$$

13. Compute the mean deviation (from the mean) for the following data:

| $x$ | 2 | 4 | 6 | 8 | 10 | 17 |
|---|---|---|---|---|---|---|
| $f$ | 1 | 6 | 10 | 7 | 2 | 2 |

*Ans.* MD = 33/14.

14. Verify the identity (where $\bar{x}$ is the mean of $x_1$ and $x_2$):

$$(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 = \tfrac{1}{2}(x_1 - x_2)^2,$$

and thus show that, for two variates,

$$|x_1 - x_2|$$

15. Verify the identity (where $\bar{x}$ is the mean of $x_1$, $x_2$, $x_3$):

$$3(x_1 - x_2)^2 + (x_1 + x_2 - 2x_3)^2 = 6\sum_1^3 (x_i - \bar{x})^2.$$

**9. The Standard Deviation of the Combination of Sets.** The following theorems involving $\sigma$ are interesting in themselves and have useful applications.

The relation $\mu_2 = \nu_2 - \nu_1^2$ is true in a more general sense than we have previously used. Its generalized meaning will be revealed in our first theorem.

**Theorem I.** *The second moment about the mean equals the second moment about an arbitrary point $P(x_0, 0)$ minus the square of the distance between the mean and $P$.*

Stated in symbols the theorem may be clearer. Suppose we have a set of $N$ variates whose mean is $\bar{x}$. Graphically, $\bar{x}$ is a point on the $x$-axis. Then if $P$ is any other point on the $x$-axis, according to Theorem I we have

$$\frac{1}{N}\sum(x - \bar{x})^2 = \frac{1}{N}\sum(x - x_0)^2 - (\bar{x} - x_0)^2. \tag{9}$$

To prove this relation we may write

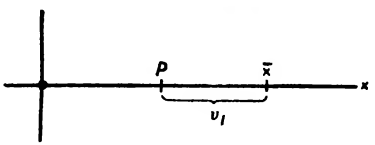
$$(x - \bar{x}) = (x - x_0) - (\bar{x} - x_0).$$

Then

$$\frac{1}{N}\sum(x - \bar{x})^2 = \frac{1}{N}\sum[(x - x_0) - (\bar{x} - x_0)]^2,$$

the right member of which simplifies into the right member of (9).

The generality of the theorem consists in extending the original definition of $\nu_2$ and $\nu_1$ so that they refer to moments about any point $P$ on the $x$-axis (except $\bar{x}$), and not merely about zero. Thus now, $\nu_2 = \frac{1}{N}\sum(x - x_0)^2$. If we take $x_0 = 0$ we have the original definition of $\nu_2$. Also, when $P$ moves back to zero, we see that $\nu_1$ becomes $\bar{x}$. In other words, the original definitions of the $\nu$'s are merely the more general definitions when zero is the value chosen for the arbitrary point. (See (1*a*) of Chapter IV.)




**Theorem II.** *The sum of the squares of deviations of the variates from their mean is less than the sum of the squares of the deviations of the variates from any other value. Therefore $\sigma$ is less than any similar "root-mean-square."*

The proof consists in showing that $\mu_2 < \nu_2$ which is left to the student as an exercise.

**Theorem III.** *Let there be one set of $n_1$ variates $x_{1i}$ ($i = 1, 2, \cdots, n_1$) and another set of $n_2$ variates $x_{2i}$ ($i = 1, 2, \cdots, n_2$) and let $\bar{x}$ be the mean of the combined sets* (Theorem VIII, Chapter III). *The variance $\sigma^2$ of the set formed by the combination of these two sets is given by the following formula:*

$$N\sigma^2 = \sum_1^{n_1}(x_{1i} - \bar{x})^2 + \sum_1^{n_2}(x_{2i} - \bar{x})^2 \tag{10}$$

where

$$N = n_1 + n_2.$$

*Proof:* The proof consists in showing that

$$\sum_1^{n_1}(x_{1i} - \bar{x})^2 + \sum_1^{n_2}(x_{2i} - \bar{x})^2 = \sum_1^{n_1+n_2}(x_i - \bar{x})^2$$

which is left as an exercise for the student.

The above theorem is not very important in itself but it is useful in proving the next theorem which gives the relation between the variance of a composite set and the variances of sub-sets.

**Theorem IV.** *Let the frequency, mean, and standard deviation be denoted by $n_1$, $\bar{x}_1$, and $\sigma_1$ for one set of variates and by $n_2$, $\bar{x}_2$, and $\sigma_2$ for a second set. The variance $\sigma^2$ of the composite set is given by the following relation:*

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2$$

where $N = n_1 + n_2$, $d_1 = \bar{x}_1 - \bar{x}$, $d_2 = \bar{x}_2 - \bar{x}$, *and $\bar{x}$ is the mean of the composite set.*

*Proof:* For the $n_1$ set, $\bar{x}$ may be regarded as an arbitrary point $P$. Hence by Theorem I we have

$$\frac{1}{n_1}\sum_1^{n_1}(x_{1i} - \bar{x}_1)^2 = \frac{1}{n_1}\sum_1^{n_1}(x_{1i} - \bar{x})^2 - (\bar{x}_1 - \bar{x})^2.$$

Multiplying through by $n_1$ this becomes

$$n_1\sigma_1^2 = \sum_1^{n_1}(x_{1i} - \bar{x})^2 - n_1d_1^2. \tag{11}$$

Similarly for the $n_2$ group we have

$$n_2\sigma_2^2 = \sum_1^{n_2}(x_{2i} - \bar{x})^2 - n_2d_2^2. \tag{12}$$

Adding (11) and (12), and using (10), we obtain

$$n_1\sigma_1^2 + n_2\sigma_2^2 = N\sigma^2 - n_1d_1^2 - n_2d_2^2.$$

Hence,

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2. \tag{13}$$

For $k$ sets combined into a single set we can generalize (13) into the following relation:

$$N\sigma^2 = \sum_1^k n_i\sigma_i^2 + \sum_1^k n_id_i^2 \tag{14}$$

where $N = \sum_1^k n_i$ and $d_i = \bar{x}_i - \bar{x}$. It is interesting to observe that $\frac{1}{N}\sum_1^k n_id_i^2$ is the variance of the means of the sub-sets. Thus we have the important relation

$$\sigma^2 = \frac{1}{N}\sum_1^k n_i\sigma_i^2 + \sigma_{\bar{x}_i}^2 \tag{14a}$$

which shows that the total variance may be broken up into two parts, one of which is the weighted mean of the variances in the sub-sets and the other is the variance of their means. These two parts are sometimes called the average variance *within* classes and the variance *between* the means of the classes. They become very important in the " Analysis of Variance " (which is explained in Part II).

COROLLARY I. *Equation (13) may be written in the following form:*

$$N\sigma^2 = n_1(\sigma_1^2 + \bar{x}_1^2) + n_2(\sigma_2^2 + \bar{x}_2^2) - N\bar{x}^2. \tag{15}$$

*Proof:* Since

$$n_1d_1^2 = n_1(\bar{x}_1 - \bar{x})^2 = n_1\bar{x}_1^2 - (2n_1\bar{x}_1\bar{x} - n_1\bar{x}^2)$$

and

$$n_2d_2^2 = n_2(\bar{x}_2 - \bar{x})^2 = n_2\bar{x}_2^2 - (2n_2\bar{x}_2\bar{x} - n_2\bar{x}^2)$$

the proof consists in showing that the sum of the terms in the end parentheses above reduces to $N\bar{x}^2$. Rearranging these terms their sum is

$$2\bar{x}(n_1\bar{x}_1 + n_2\bar{x}_2) - \bar{x}^2(n_1 + n_2),$$

which by Theorem VIII (Chapter III) becomes

$$2\bar{x}N\bar{x} - \bar{x}^2N = N\bar{x}^2.$$

Generalizing for $k$ groups, (15) becomes

$$N\sigma^2 = \sum_1^k n_i(\sigma_i^2 + \bar{x}_i^2) - N\bar{x}^2. \tag{16}$$

COROLLARY II. *Equation (13) may also be written in the form:*

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + \frac{n_1n_2}{N}(\bar{x}_1 - \bar{x}_2)^2. \tag{17}$$

The proof consists in showing that

$$n_1d_1^2 + n_2d_2^2 = \frac{n_1n_2}{N}(\bar{x}_1 - \bar{x}_2)^2.$$

This is left as an exercise.

For purposes of computation, (17) may be more convenient than either (13) or (15) because it does not require $\bar{x}$, but it does not lend itself to a generalization for $k$ sets. Generalizations may be useful both for computing and for theoretical purposes. Formula (14) is particularly useful in developing the theory of a later section.

For convenience, the formulas of Theorem VIII, Chapter III, are repeated here:

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}, \tag{18}$$

$$\bar{x} = \frac{1}{N}\sum_1^k n_i\bar{x}_i,\ N = \sum_1^k n_i. \tag{18a}$$

**Theorem V.** *Consider $k$ sets. Suppose the second moment of each set is taken about the mean, $\bar{x}$, of the combined sets. Let $\nu_2^{(i)}$ represent this moment for the $i$th set. Then the variance $\sigma^2$ for the combined sets is given by*

$$N\sigma^2 = n_1\nu_2^{(1)} + n_2\nu_2^{(2)} + \cdots + n_k\nu_2^{(k)} = \sum_1^k n_i\nu_2^{(i)} \tag{19}$$

when $n_i$ represents the frequency in the $i$th set and $\sum_1^k n_i = N$.

*Proof:* We may write (10) in the form

$$N\sigma^2 = n_1\nu_2^{(1)} + n_2\nu_2^{(2)}.$$

So, generalizing this form of (10) for $k$ sets, we obtain (19).

The next theorem gives the standard deviation of the distribution formed by the first $N$ integers, that is, when $x = 1, 2, 3 \cdots, N$. It is

useful in cases when the variates are recorded not by measurements but by their respective positions when ranked in order with respect to some character or property.

**Theorem VI.** *The standard deviation $\sigma$ of the first $N$ natural numbers is given by*

$$\sigma = \left[\frac{(N^2 - 1)}{12}\right]^{1/2}. \tag{20}$$

*Proof:* By a fundamental definition we have

$$\sigma^2 = \frac{1}{N}\sum_{x=1}^{N} x^2 - \left[\frac{1}{N}\sum_{x=1}^{N} x\right]^2$$

and by Theorems IV and V of Chapter III, this becomes

$$\sigma^2 = \tfrac{1}{6}(N + 1)(2N + 1) - \tfrac{1}{4}(N + 1)^2$$

which reduces to

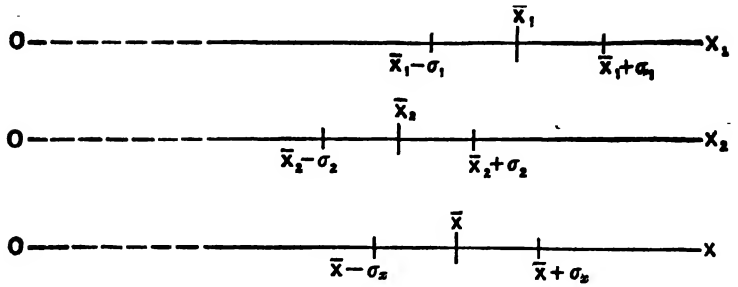
$$\sigma^2 = \frac{N^2 - 1}{12},$$

whence we obtain (20).

**10. Graphical Representation.** We have shown that, if certain statistics are given for two sub-sets,

| | | | |
|---|---|---|---|
| *Sub-sets* | $n_1$ | $\bar{x}_1$ | $\sigma_1$ |
| | $n_2$ | $\bar{x}_2$ | $\sigma_2$ |
| *Composite set* | $N$ | $\bar{x}$ | $\sigma_x$ |

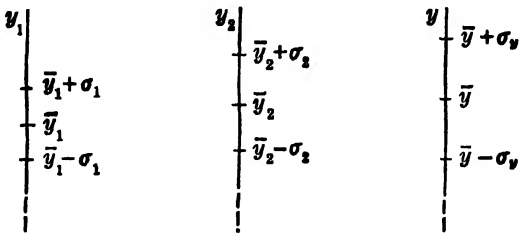
the corresponding statistics for the composite set may be obtained by means of (13) and (18$a$). We have been thinking of these statistics as relating to distributions in the $x$-direction. The following diagrams show how the means and standard deviations of three such distributions may be represented geometrically by the points whose ordinates are zero and whose abscissas are, respectively, $\bar{x}_1$, $(\bar{x}_1 \pm \sigma_1)$; $\bar{x}_2$, $(\bar{x}_2 \pm \sigma_2)$; and $\bar{x}$, $(\bar{x} \pm \sigma_x)$. The points are plotted on three different axes to avoid confusion, but they are to be thought of as being referred to the same origin and plotted on the same scale.




It should be clear that Theorems I–IV (§9) will apply to distributions in the $y$-direction as well as in the $x$-direction. In particular, it is obvious that (13) and (18$a$) hold if we replace $x$ by $y$. Then the graphical representation of the means and standard deviations

| *Sub-sets* | | *Composite set* |
|---|---|---|
| $n_1$ | $n_2$ | $N$ |
| $\bar{y}_1$ | $\bar{y}_2$ | $\bar{y}$ |
| $\sigma_1$ | $\sigma_2$ | $\sigma_y$ |

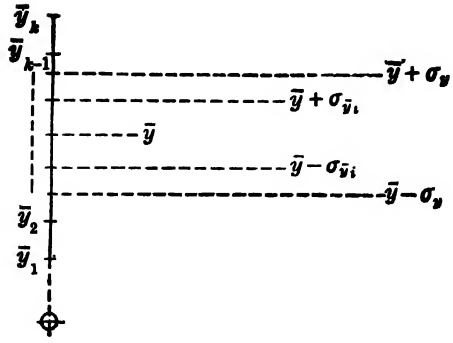
is shown below.




It will be helpful to discuss one more notion in this connection. Suppose the $y$ composite set is made up of $k$ sub-sets and the means $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_k$, of these sub-sets are plotted on the $y$-axis as shown by the labels on the left side of the axis in the figure on page 105.

We will denote the standard deviation of these means by $\sigma_{\bar{y}_i}$. Then the points $\bar{y}$, $(\bar{y} \pm \sigma_{\bar{y}_i})$, and $(\bar{y} \pm \sigma_y)$, may be plotted as shown. We would expect less variability among the means of the sub-sets than among the $y$'s of the composite set, that is, that $\sigma_{\bar{y}_i}$ would be

less than $\sigma_y$. It is clear that (14) and (14$a$) hold when $x$ is replaced by $y$.




A grasp of these notions will help in the analysis of Table 27 which the student is asked to make in problems 5 and 6 below.

## Exercises

**1.** (*a*) Show that $\nu_1 = \frac{1}{N}\sum(x_i - x_0) = (\bar{x} - x_0)$.

(*b*) Derive equations (9) and (13). If $n_1 = n_2$, what does (13) reduce to?

**2.** Given the following information about two sets of data:

| I | II |
|---|---|
| $n_1 = 20$ | $n_2 = 30$ |
| $\bar{x}_1 = 25$ | $\bar{x}_2 = 20$ |
| $\sigma_1^2 = 5$ | $\sigma_2^2 = 4$. |

Find the mean and variance of the composite set.

**3.** Think of the two groups in Exercise 2, page 97, as combined into a single set.

(*a*) Find the mean of the combined set by formula (18).

(*b*) Find the standard deviation of the combined set using result of (*a*) and formula (13). *Ans.* $\bar{x} = 68.72$, $\sigma = 12.45$.

**4.** Using Theorem VI find the mean and standard deviation of the first 25 natural numbers.

**5.** Consider Table 27. Observe that the first and last columns form a frequency distribution and that columns (1) to (8) are subdistributions whose totals add up to $N = 260$ which is also the sum of the last column. Let $n_i$ represent the frequency in the $i$th column and answer the following questions: $n_1 = ?$, $n_4 = ?$, $n_8 = ?$, $\sum_1^8 n_i = ?$ Let $\bar{y}_i$ and $\sigma_i^2$ represent mean and variance in the $i$th column. Find the mean and variance of each of the columns (1) to (8), first in $v$ units where $v = (y - 85)/10$. Check your answers with those given at the bottom of the table.

TABLE 27

| $y$ | $v$ | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | $f$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 125 | 4 | | | | | 2 | 3 | 2 | | 7 |
| 115 | 3 | | | 1 | 3 | 1 | 4 | 4 | 4 | 17 |
| 105 | 2 | | | 5 | 7 | 8 | 11 | 8 | 7 | 46 |
| 95 | 1 | | 2 | 1 | 10 | 12 | 9 | 8 | 2 | 44 |
| 85 | 0 | 1 | 3 | 12 | 11 | 7 | 12 | 7 | 1 | 54 |
| 75 | −1 | 2 | 1 | 5 | 6 | 16 | 8 | 5 | | 43 |
| 65 | −2 | 2 | 5 | 5 | 8 | 8 | 6 | 1 | | 35 |
| 55 | −3 | 2 | 3 | 3 | 4 | 1 | 1 | | | 14 |
| | $n_i$ | 7 | 14 | 32 | 49 | 55 | 54 | 35 | 14 | $N = 260$ |
| Answers | $\bar{y}_i$ | 67.86 | 72.14 | 81.87 | 84.80 | 85.73 | 90.92 | 95.57 | 105.0 | $\bar{y} = 87.31$ |
| | $\sigma_i^2$ | 106.12 | 191.83 | 246.48 | 283.63 | 257.65 | 294.51 | 222.53 | 71.43 | $\sigma_y^2 = 303.11$ |

6. Using formulas (18*a*) and (14) find the mean $\bar{y}$ and variance $\sigma_y^2$ of the total distribution in Table 27 and check your answers with those given at the bottom of the last column.

*Hint.* The student will observe that the means, $\bar{y}_i$, of the columns in Table 27 are the values denoted by $y$ in Exercise 3, page 97. The weighted mean of these mean values is the mean of the whole table. That is, from (18*a*),

$$\bar{y} = \frac{1}{N}\sum_1^k n_i\bar{y}_i$$
$$= 87.31.$$

The answer 56.66 (Exercise 3) is the variance, $\sigma_{\bar{y}_i}^2$, of the *means* of the columns of Table 27 and is not to be confused with the variance $\sigma_y^2$ of the whole table. In using (14), $\sigma^2$ is the variance of the whole table, $\sigma_i^2$ is the variance of the $i$th column, and the expression $\sum_1^k n_i d_i^2$ equals $N\sigma_{\bar{y}_i}^2$ where $\sigma_{\bar{y}_i}^2$ is the variance of the means of the columns since now $d_i = \bar{y}_i - \bar{y}$.

7. In Theorem V (§9) show that

$$\nu_2^{(i)} = \sigma_i^2 + d_i^2.$$

Hence prove that (19) may be derived from (14) by showing that (14) may be written as follows:

$$N\sigma^2 = \sum_1^k n_i(\sigma_i^2 + d_i^2).$$

8. (*a*) Derive the following relation from (18*a*),

$$\bar{x}_1 = \frac{1}{n_1}\left[N\bar{x} - \sum_{i=2}^k n_i\bar{x}_i\right].$$

What does this formula become when $k = 2$?

(*b*) Derive the following relation from (15),

$$\sigma_1^2 = \frac{1}{n_1}\left[N(\sigma^2 + \bar{x}^2) - n_2(\sigma_2^2 + \bar{x}_2^2)\right] - \bar{x}_1^2.$$

9. In a certain distribution of $N = 25$ measurements it was found that $\bar{x} = 56$ inches and $\sigma = 2$ inches. After these results were computed it was discovered that a mistake had been made in one of the measurements which was recorded as 64 inches. Find the mean and standard deviation if the incorrect variate, 64, is omitted.

*Hint.* Let $n_1 = 24$, $n_2 = 1$. Then $\bar{x}_2 = 64$ and $\sigma_2 = 0$. To find $\bar{x}_1$ and $\sigma_1$ use formulas in Exercise 8 above.

10. If two or more variates are deleted from a distribution for which $N$, $\bar{x}$, and $\sigma$ are given, show how to compute the mean and variance of the remaining variates.

**11.** Consider a composite set consisting of $k$ sub-sets and let $\sigma_i^2$ and $n_i$ denote, respectively, the variance and number of variates in the $i$th sub-set, and $N = \sum_1^k n_i$.

(*a*) If the sub-sets have equal means, show that the variance of the composite set is given by

$$\sigma^2 = \frac{1}{N}\sum_1^k n_i\sigma_i^2.$$

(*b*) If the sub-sets each contain the same number of variates and have equal means, show that

$$\sigma^2 = \frac{1}{k}\sum_1^k \sigma_i^2.$$

# CHAPTER VI

## TYPES OF DISTRIBUTIONS. THE NORMAL CURVE

**1. Skewness and Kurtosis.** The shapes of frequency distributions are not all alike. Unimodal distributions may differ in two ways with respect to form. These differences can be described more easily if we think in terms of frequency curves. The curve may be quite

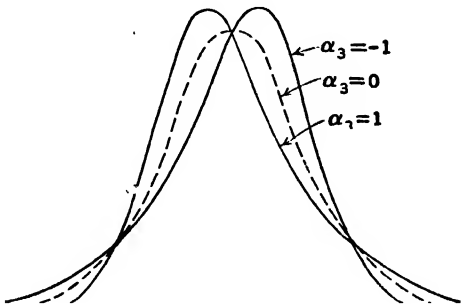



FIG. 18

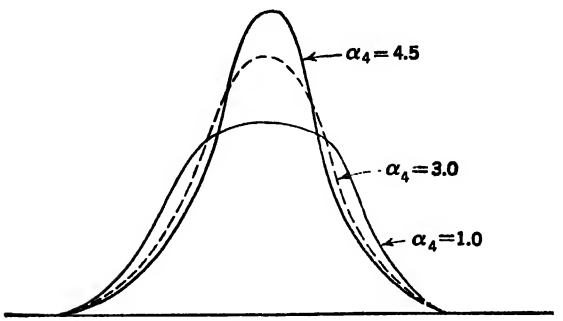



FIG. 19

symmetrical, or it may be skew, bulging out on one side more than on the other. Secondly, the top of the curve may be narrow and peaked, or it may be somewhat flat giving a mound-shape effect.

The mean and standard deviation are not sufficient to detect these characteristics, so we need other measures to describe them. Con-

TABLE 28

| | A | B | C |
|---|---|---|---|
| $u$ | $f$ | $f$ | $f$ |
| −3 | 0 | 1 | 0 |
| −2 | 3 | 1 | 1 |
| −1 | 6 | 5 | 10 |
| 0 | 7 | 11 | 6 |
| 1 | 6 | 5 | 5 |
| 2 | 3 | 1 | 2 |
| 3 | 0 | 1 | 1 |
| Sums | 25 | 25 | 25 |

sider, for example, the three distributions of the weights (in class units)[1] of different breeds of mice 120-130 days old given in Table 28. Experiments on mice are important in cancer research. These distributions are, however, somewhat fictitious, being adapted from some actual data for purposes of illustration.

The student may easily verify that for each of these distributions we find the same mean and standard deviation, namely,

$$\bar{u} = 0, \qquad \sigma_u = 1.2.$$

One may see from their histograms that these distributions are essentially different in shape even though they all have the same mean and standard deviation. These differences would be more pronounced if $N$ were so large that the shapes approached a regular and smooth form. Such a large value is called the "population" or "universe" and the value of $N$ that we usually have at hand is a "sample."

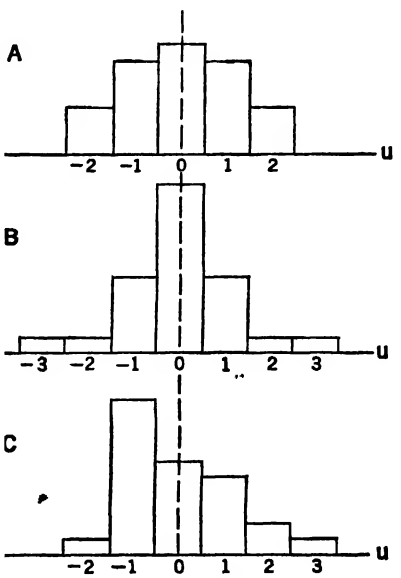


FIG. 20

[1] Neither the original units nor the class interval need concern us here.

Lack of symmetry in a distribution is known as "skewness." This characteristic is measured by $\alpha_3$. If a distribution is symmetrical $\alpha_3 = 0$, but $\alpha_3$ may be positive or negative depending upon whether the long tail of the distribution extends to the right or the left of the mean. (See Figure 18.)

Figure 19 exhibits curves with different degrees of flatness or peakedness. The flatness that we are now describing is in the neighborhood of the mode and is not to be confused with the flatness of a curve as a whole which is due to spread or dispersion. The curves in Figure 19 all have the same spread. So their flatness depends upon the relative amount of material in the vicinity of the mode. This characteristic of a curve is called "kurtosis" and is measured by $\alpha_4$. By the calculus it can be demonstrated that $\alpha_4 = 3$ for a certain type of distribution which is called the normal curve. A frequency curve is said to have positive kurtosis if $\alpha_4 > 3$ and negative kurtosis if $\alpha_4 < 3$. It seems, however, that any combination of kurtosis and peakedness may occur.[1] The values of $\alpha_3$ and $\alpha_4$ computed for an observed distribution are useful in selecting the curve which will best represent the type to which that distribution belongs.

Both $\alpha_3$ and $\alpha_4$ are abstract numbers and therefore skewness and kurtosis in different distributions may be compared by these measures. Therefore our definitions are

$$(1) \qquad \begin{cases} \alpha_3 \text{ is a measure of skewness,} \\ \alpha_4 \text{ is a measure of kurtosis.} \end{cases}$$

For an unsymmetrical distribution the distance between the mean and mode may be used to measure the degree of asymmetry or skewness, because the mean and mode coincide in a symmetrical distribution. Since we wish any measure of skewness to be a pure number, we would express this distance in units of the standard deviation, thus (mean − mode)/$\sigma$. Now it happens that there is a certain curve known as Pearson's Type III which is used to represent certain

[1] A Common Error Concerning Kurtosis — I. Kaplansky. *J. Amer. Stat. Assoc.*, vol. 40, p. 259, June 1945. In this connection, Professor I. W. Burr comments: "The shape of the hump of the curve has less influence on $\alpha_4$ than does the length of the tails. In Figure 19, the curve with $\alpha_4 = 4.5$ should have the longest tails."

skew distributions, and it can be shown by higher mathematics that, for this curve,

$$\frac{\text{mean} - \text{mode}}{\sigma} = \frac{\alpha_3}{2}. \tag{2}$$

So this relation[1] may be used as a formula for obtaining the approximate mode.

### Exercise

Find $\alpha_3$ and $\alpha_4$ for each of the distributions $A$, $B$, and $C$, in Table 28.

**2. Frequency Curves.** As the student extends his experience he finds several types of distributions. It is important in certain problems to differentiate between them. Differences in type lead to the study of frequency curves. There are several standard curves to represent the different types of distributions that arise in practical statistics.[2] Each of these is specified by a mathematical function $y = f(x)$ where $f(x)$ is a general symbol for any function of $x$. It is, of course, a different expression for each of the different curves.

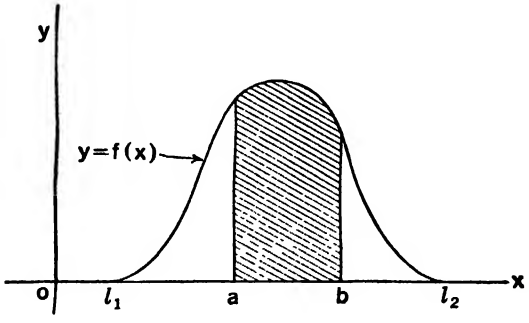



FIG. 21

Such functions are also called distribution functions. A complete discussion of this subject belongs to the field of advanced statistics. However, there are some simple concepts relating to frequency curves which will be useful in our work.

If a frequency curve is used to represent a given distribution, the total area under the curve corresponds to the total frequency $N$,

[1] Because of this relation some writers use $\alpha_3/2$ as a measure of skewness instead of $\alpha_3$. Also some authors adopt a different convention as to sign, defining skewness as negative when the mean is greater than the mode.

[2] See Chapter III, Part II.

and therefore the partial area under the curve between the ordinates erected at $x = a$ and $x = b$ (Figure 21) represents the number of variates with measurement or character between $a$ and $b$. The limits between which the theoretical distribution ranges are denoted by $l_1$ and $l_2$. It is often convenient and causes no loss of generality to suppose that the total area under the curve is unity or 100%, in which case the partial area between $a$ and $b$ represents the percentage of variates having the given character.

In mathematical language the "area under $f(x)$ between $a$ and $b$" is called the "integral of $f(x)$ from $a$ to $b$," and is denoted by the symbol

$$\int_a^b f(x)\,dx.$$

However, we will abbreviate this symbol and use merely $\int_a^b$ to denote such an area.

Without attempting to be rigorous, we may say that the total area under the curve is the limit of the area of the appropriate histogram whose rectangles have bases $\Delta x$ and altitudes $f(x)$, as $\Delta x$ is taken smaller and smaller and approaches zero. Thus

$$\int f(x)\,dx = \lim_{\Delta x \to 0} \sum f(x)\,\Delta x.$$

The integral sign $\int$ is a conventionalized $S$ and denotes the sum of elements of area with bases $dx$ and altitudes $y = f(x)$. The letters written at the top and bottom of $\int$ denote the range over which the sum is to be taken. Therefore the notation $\int_a^b y\,dx$ or $\int_a^b f(x)\,dx$ represents the area which is bounded by the curve $y = f(x)$, the ordinates at $x = a$ and $x = b$, and the $x$-axis. (Figure 21.)

The integral of $y = f(x)$ from $l_1$ to $l_2$ denotes the total frequency $N$. Therefore,

$$N = \int_{l_1}^{l_2}.$$

Hence, the proportion of variates having some character $x$, such that $a \leq x \leq b$, is given by $\frac{1}{N}\int_a^b$. If $N$ is taken as unity or 100%, then

$\int_a^b$ denotes the percentage of variates having the given character. The integral represented by this symbol also denotes the probability that a variate chosen at random from the universe $y = f(x)$ will have a value between $a$ and $b$.

**3. The Normal Curve.** Perhaps the most important of all frequency curves is the so-called normal[1] curve whose equation may be written

$$y = Ke^{-h^2(x-m)^2}, \tag{3}$$

where $K$, $h^2$, and $m$ represent numbers whose significance will be explained presently. The curve is bell-shaped and is symmetrical about the line $x = m$. It was first discovered by a famous French mathematician, De Moivre, over two hundred years ago and published in 1733. He obtained it while working on certain problems in games of chance which were proposed to him by the gamblers of his day. Because of this origin and because the data from certain coin- and dice-throwing experiments closely approach it in form, it is often called the normal probability curve. Actual statistical use of the normal curve began with the work of the famous mathematical astronomers, Laplace (1749–1827) and Gauss (1777–1855), each of whom derived it independently and presumably without knowing of De Moivre's treatment.[2] They found that it represented very well the errors of observation in the physical sciences. For this reason it has been called the normal curve of error, where error is used in the sense of a deviation from the true value. Since that time experience has shown that it serves quite well to describe many of the distributions which arise in the fields of biology, education, and sociology. Much of the theory of statistics is built around it.

The calculus is required to define the moments of a theoretical distribution specified by a frequency curve $y = f(x)$. (These definitions are given in Part II.) It turns out that *the mean of the distribution specified by (3) is m and its variance is* $1/(2h^2)$. The constant $K$ is determined so that the area under the curve shall have some relevant value. In describing an observed distribution by

[1] The term " normal " used here should not be interpreted to mean that other types of distribution are abnormal.

[2] For a more extensive history see (*a*) "Bi-centenary of the Normal Curve," *Jour. Amer. Statistical Assoc.*, vol. 29 (1934), pp. 72–75. (*b*) "Mathematical Statistics" (Carus Monograph) — Rietz, Ch. 3.

means of a normal curve, we wish to have the number of area units under the curve (3) equal to the number $N$ of observed variates. When this condition is imposed, $K = Nh/\sqrt{\pi}$ and we see that $K$ depends also on $h$. If we adopt the same notation[1] here as we used for an observed distribution, we have

$$m = \bar{x}, \quad h^2 = \frac{1}{2\sigma_x^2}, \quad K = \frac{N}{\sigma_x\sqrt{2\pi}}.$$

Upon making these replacements, (3) becomes

$$y = \frac{N}{\sigma_x\sqrt{2\pi}} e^{-(x-\bar{x})^2/2\sigma_x^2}. \tag{3a}$$

**4. Standard Form.** The letters $\pi$ and $e$ represent numbers which always have the same values (see §1, Chapter I). But each of the letters $m$, $h$, and $K$ may take on different values in different situations. Such constants are called parameters, and (3) really represents a *family* of curves. Similarly, in (3a), $\bar{x}$, $\sigma$, and $N$ are parameters. For assigned values they determine, respectively, the position of the curve along the $x$-axis, its steepness, and its "size" but they do not have anything to do with its fundamental characteristics (*i.e.*, those properties which differentiate it from all other curves). In order to study these characteristic properties it is convenient to represent the curve by an equation which will be independent of the parameters; in other words, to eliminate them from the equation by a transformation. This is accomplished by considering the total area under the curve as unity, taking the origin at the mean, and using the standard deviation as the unit of horizontal measurement. In mathematical language this means that we set $N = 1$, and $t = (x - \bar{x})/\sigma_x$. We will denote the resulting function by $\phi(t)$, that is,

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \tag{4}$$

which is called the *standard form* of the normal curve.

*A variable, $t$, which is distributed in accord with (4) is said to be normally distributed with mean zero and unit standard deviation.*

Just as coördinates of points on the curve are denoted by $(x, y)$

[1] In the theory of sampling, Part II, it is necessary to distinguish the moments of a sample from those of the parent universe by the use of different symbols.

in the case of equation (3$a$), so in equation (4) $t$ refers to abscissas and $\phi(t)$ refers to ordinates. The relation between the two systems of coördinates is given by

$$x = t\sigma + \bar{x} \tag{5}$$

for abscissas, and

$$y = \frac{N}{\sigma}\phi(t) \tag{6}$$

for ordinates. Equation (6) follows from (3$a$) and (4). If the area under the curve is taken as unity, then $y = \frac{1}{\sigma}\phi(t)$, that is, $\phi(t) = \sigma y$. This says that since the abscissas are compressed by $\sigma$ in changing from arbitrary units into standard units, so the ordinates must be stretched by $\sigma$ if the area under the curve is to be the same in the two scales of measurement.

**5. Tables of Standard Ordinates and Areas.** One of the reasons for writing the equation in standard form is that the ordinates and

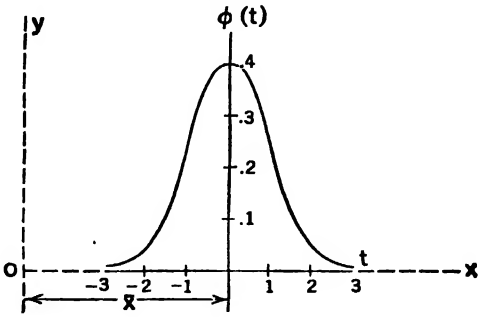


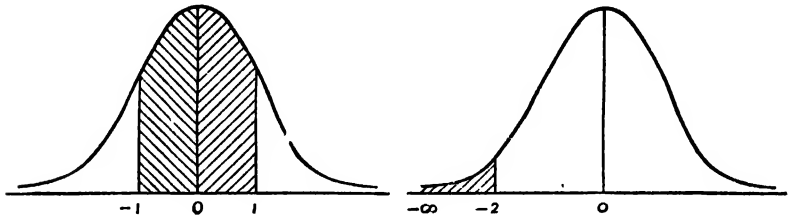

FIG. 22

areas may be tabulated once and for all. These tables are given in the Appendix. We see from (4) that $\phi(-t) = \phi(+t)$, *i.e.*, the ordinates for negative values of $t$ are the same as for the corresponding positive values of $t$, and the curve is symmetrical about the ordinate at $t = 0$. Therefore it is necessary to tabulate values of $\phi(t)$ for positive $t$'s only. Equation (4) may be graphed by plotting the points corresponding to a few well chosen values from the tables and drawing a smooth curve through them. (Figure 22.)

The curve approaches very close to the horizontal axis at each extremity but is asymptotic, that is, it does not quite touch the axis

no matter how far extended. We say its limits are at $-\infty$ and $+\infty$. Although the infinite abscissal range is never met in practice it may be characteristic of the "universe" from which a given distribution is a sample. Therefore, this infinite feature is useful in theoretical investigations. Moreover, even in representing observed distributions the infinite range causes no practical difficulty because the curve comes down to the horizontal axis very rapidly beyond $t = \pm 3$. The combined area at each extremity beyond $t = \pm 3$ is only .27 of 1% of the total area under the curve.

Partial areas between ordinates erected at various values of $t$, say between $t = a$ and $t = b$, are denoted by $\int_a^b$. Thus the area from




$t = 0$ to $t = 1$ is given by $\int_0^1 = .3413$. (See Table I, Appendix.) Since the total area under $\phi(t)$ is taken as unity the area on either side of $t = 0$ is 0.5 and it is only necessary to tabulate the areas $\int_0^t$ for positive values of $t$. Thus the area from $t = -1$ to $t = 0$ is equal to the area from $t = 0$ to $t = 1$. In symbols this would be stated as follows:

$$\int_{-1}^{0} = \int_0^1.$$

Any other areas required may be found by an appropriate addition or subtraction of tabular values. For example, suppose the area below $t = -2$ is required. This is denoted by $\int_{-\infty}^{-2}$. Now the area from $-\infty$ to $-2$ equals 0.5 minus the area from $-2$ to 0. And the area from $-2$ to 0 is the same as from 0 to 2. That is,

$$\int_{-\infty}^{-2} = .5 - \int_{-2}^{0}. \quad \text{But} \quad \int_{-2}^{0} = \int_0^2 = .4772.$$

$$\therefore \int_{-\infty}^{-2} = .5 - .4772 = .0228.$$

Both areas and ordinates for decimal values of $t$ between tenths may be approximated by interpolating between the values given in the tables.

The illustrative examples following §6 will help the student become familiar with the tables. He should verify the answers and draw a simple sketch of the curve showing the ordinates or areas in each case.

The symbol $\int_{-\infty}^{t}$ denotes a cumulative relative frequency, *i.e.* the percentage of the total frequency $N$ which is less than $t$. In order to find values of $\int_{-\infty}^{t}$ from the tables, for assigned values of $t$, the student should observe (from a figure) that

$$\int_{-\infty}^{t} = .5 \pm \int_{0}^{t}$$

the plus or minus sign to be used according as $t$ is positive or negative.

**6. Properties.** A knowledge of the properties of the normal curve is essential for an intelligent use of the curve in practical statistics. A demonstration of some of these properties is beyond the scope of the present discussion although quite simple in the calculus. The following properties are the most important and interesting.

1. The mean, median, and mode coincide at $t = 0$. The height of the maximum ordinate in standard form is $1/\sqrt{2\pi}$ because when $t = 0$, $\phi(t) = 1/\sqrt{2\pi} = .3989$.

2. Since the standard deviation is the unit of measurement along the horizontal axis, $\sigma_t = 1$ in the $t$ scale. Any $t$ value may be converted into the corresponding $x$ value by (5). In the vertical direction $N/\sigma$ is the unit of measurement and any $\phi(t)$ ordinate may be converted into $y$ units by means of (6).

The area under (3) in the range from $x = c$ to $x = d$ is denoted by

$$\int_{c}^{d} y\,dx.$$

If $t = a$ and $t = b$ denote the corresponding range in standard units, then

$$\int_{a}^{b} \phi(t)\,dt \tag{7}$$

denotes the corresponding area, in standard units, under (4). It is

shown in the calculus that $dx = \sigma_x dt$. Therefore from (6) we have

$$\int_c^d y\,dx = N\int_a^b \phi(t)\,dt. \tag{8}$$

If the interval goes from $x = c$ to $x = d$, (8) says that

$$\textit{Frequency over } (c, d) = N\int_a^b \tag{9}$$

where

$$a = (c - \bar{x})/\sigma_x,\ b = (d - \bar{x})/\sigma_x. \tag{10}$$

This merely means that the percentages (relative frequencies) obtained from the tables may be converted into numbers (frequencies) by multiplying the percentages by $N$.

3. The curve changes from concave to convex at $t = \pm 1$. In the $x$-scale, referred to the origin of $x$, these points are at $x = \bar{x} \pm \sigma_x$. They are called points of inflection and their position is important in making an accurate drawing of the curve.

4. The standard deviation is approximately 25% greater than the mean deviation. More precisely, MD $= \sigma\sqrt{\frac{2}{\pi}} = .798\sigma$. $\left(\sqrt{\frac{\pi}{2}} = 1.2533.\right)$

5. The quartiles, $Q_1$ and $Q_3$, are equidistant from $t = 0$ and therefore from the mean. By definition

$$Q_3 \text{ is that value of } t \text{ for which } \int_{-\infty}^{t} = .75,$$

*i.e.*, for which $\int_0^t = .25$. From the tables this is $t = .6745$. Therefore in arbitrary units,

$$Q_3 = \bar{x} + .6745\sigma_x \textit{ and } Q_1 = \bar{x} - .6745\sigma_x.$$

6. The quartile deviation (semi-interquartile range) for a normal distribution will be denoted by $E$. Its value is

$$E = \frac{Q_3 - Q_1}{2} = \frac{(\bar{x} + .6745\sigma) - (\bar{x} - .6745\sigma)}{2} = .6745\sigma.$$

In standard units this is $s = E/\sigma = .6745$.

7. The quantity $E$ (or $s$) has a significance in probability theory. If a variable $x$ is distributed according to the normal curve, the

probability is one half that a variate selected at random will have a value between $\bar{x} - E$ and $\bar{x} + E$. The reason for this statement is that 50% of the variates have values within this range. $E$ is commonly, though somewhat ambiguously, called "probable error."

8. $E$ is in units of $x$ whereas $s$ is a value of $t$, that is, $s$ is the value $t = .6745$, and $E$ is the value $x = .6745\sigma_x$. Just as $\sigma_x$ may be used as a yardstick in scaling off a distribution on either side of the mean (§6, Chapter V), so may $E$ or $s$ be used in a similar manner. When thinking of them in this way it is useful to regard $E$ as a yardstick about two-thirds the length of $\sigma_x$. The following table gives the end-points of certain intervals in $t$, $x'$, and $x$ units, respectively, where $t = x'/\sigma_x$ and $x' = x - \bar{x}$.

*End Points of Certain Intervals in t, x', x*

| *When σ is the unit* | | | *When E is the unit* | | |
|---|---|---|---|---|---|
| $t$ | $x'$ | $x$ | $t$ | $x'$ | $x$ |
| 0 | 0 | $\bar{x}$ | 0 | 0 | $\bar{x}$ |
| $\pm 1$ | $\pm\sigma$ | $\bar{x} \pm \sigma$ | $\pm .6745$ | $\pm .6745\sigma$ | $\bar{x} \pm .6745\sigma$ |
| $\pm 2$ | $\pm 2\sigma$ | $\bar{x} \pm 2\sigma$ | $\pm 1.349$ | $\pm 1.349\sigma$ | $\bar{x} \pm 1.349\sigma$ |
| $\pm 3$ | $\pm 3\sigma$ | $\bar{x} \pm 3\sigma$ | $\pm 2.023$ | $\pm 2.023\sigma$ | $\bar{x} \pm 2.023\sigma$ |

The percentage distribution of area under the normal curve is given (approximately) in Figure 23 where $\sigma_x$ is the unit of measurement along the horizontal axes and in Figure 24 where $s$ is the unit. The percentages given in the figures may be regarded as abridged tables. Of course the tables in the Appendix will ordinarily be used in problems.

With reference to Figure 23, it is sometimes said that if values of $x$ are normally distributed, the probability that a value chosen at random will fall within the range $x_1 \leq x \leq x_2$, where $x_1 = \bar{x} - \sigma_x$ and $x_2 = \bar{x} + \sigma_x$, is .68.

9. Astronomers and physicists have called $h$ the "modulus of precision." From the relation $h = 1/(\sqrt{2}\sigma)$, it is evident that $h$ increases as $\sigma$ decreases. And as $h$ increases, the curve (with $N$ and $m$ kept constant) becomes narrower in the neighborhood of $m$ and in this sense $h$ measures the closeness of the values of $x$ to their mean.

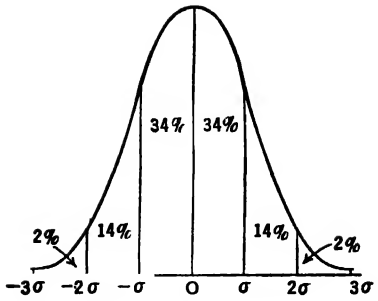



FIG. 23

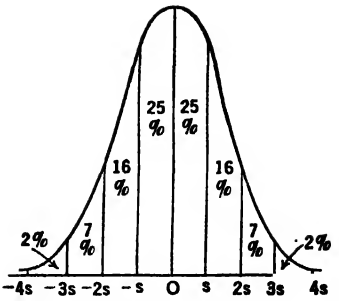



FIG. 24

10. The curve is symmetrical and $\alpha_3 = 0$. The fourth moment about the mean is equal to three times the square of the second moment about the mean, *i.e.*, $\mu_4 = 3\mu_2^2$ and therefore $\alpha_4 = \mu_4/\mu_2^2 = 3$.

## Examples

**1.** Find the ordinates of $\phi(t)$ for (*a*) $t = 2.3$, (*b*) $t = -2.3$, (*c*) $t = .67$.

*Solutions* from the tables in the Appendix:

(*a*) $\phi(2.3) = .02833$
(*b*) $\phi(-2.3) = .02833$
(*c*) $\phi(.67) = .31874$

**2.** Find the following areas under $\phi(t)$ and use the integral notation:

(*a*) From $t = 0$ to $t = 3.00$
(*b*) From $t = 1.5$ to $t = 2.5$
(*c*) From $t = -2$ to $t = 1.3$
(*d*) From $t = 0$ to $t = 0.6745$

*Solutions* from the tables:

(*a*) The required area is given by $\int_0^3$ which we find to be .49865.

(*b*) The area from $t = 0$ to $t = 1.5$ is $\int_0^{1.5} = .43319$, and from $t = 0$ to $t = 2.5$ is $\int_0^{2.5} = .49379$. Therefore the required area is $\int_{1.5}^{2.5} = \int_0^{2.5} - \int_0^{1.5} = .0606$.

(*c*) Since the area from $t = 0$ to $t = -2$ is the same as from $t = 0$ to $t = +2$ we have

$$\int_{-2}^{1.3} = \int_0^2 + \int_0^{1.3} = .47725 + .40320 = .88045.$$

(*d*) Here we must interpolate:

For $t = .67$, $\int_0^t = .24857$

For $t = .6745$, $\int_0^t = A$ (say)

For $t = .68$, $\int_0^t = .25175.$

Therefore

$$\frac{A - .24857}{.25175 - .24857} = \frac{.0045}{.01}$$

whence

$$A = .25.$$

**3.** Show that for equation (3), the percentages of area outside the given ranges are as stated below:

$$\begin{aligned} &\text{Above } \bar{x} + \sigma = 15.87\% \\ &\text{Outside } \bar{x} \pm \sigma = 31.74\% \\ &\text{Outside } \bar{x} \pm 2\sigma = 4.56\% \\ &\text{Outside } \bar{x} \pm 3\sigma = 0.27\% \end{aligned}$$

*Solution:* Converting these ranges into $t$ units, and remembering that only the positive half of the area under $\phi(t)$ is tabulated and equals .5, we have

$$\text{Area above } t = 1 \text{ is } .5 - \int_0^1 = .1587 = 15.87\%$$

$$\text{Area outside } t = \pm 1 \text{ is } 2(15.87\%) = 31.74\%$$

$$\text{Area outside } t = \pm 2 \text{ is } 2\left(.5 - \int_0^2\right) = .0456 = 4.56\%$$

$$\text{Area outside } t = \pm 3 \text{ is } 2\left(.5 - \int_0^3\right) = .0027 = 0.27\%$$

**4.** Given $N = 1500$, $\bar{x} = 75$, $\sigma_x = 10$. If the variates are distributed according to the normal curve, (*a*) find the value of $x$ for which *cum* $f = 800$, (*b*) for which *cum* $f = 450$, (*c*) how many of the $N$ variates lie where $x < 80$?

*Solutions:*

(*a*) By definition, *cum* $f = \int_{-\infty}^{x}$

and from (8), $\int_{-\infty}^{x} = N \int_{-\infty}^{t}$

$$\therefore 800 = 1500 \int_{-\infty}^{t}$$

*i.e.*, $$\int_{-\infty}^{t} = .5333.$$

But $$\int_{-\infty}^{t} = .5 + \int_{0}^{t}$$

$$\therefore \int_{0}^{t} = .0333$$

whence from the tables,

$$t = .083.$$

Substituting in equation (5),

$$x = 75.83.$$

(*b*) We have $\int_{-\infty}^{t} = 45/150 = .3$ and $t$ is negative.

Since $$\int_{-\infty}^{t} = .5 - \int_{0}^{t}$$

we have $$\int_{0}^{t} = .2$$

whence we find that

$$t = -.524$$

so

$$x = 69.76.$$

(*c*) From the relation $t = (x - \bar{x})/\sigma_x$ we find that

$$t = .5 \quad \text{when} \quad x = 80.$$

From the tables,

$$\int_{-\infty}^{.5} = .69146.$$

From (8) we have

$$\int_{-\infty}^{x} = 1500\ (.69146)$$

$$= 1037.2.$$

## Exercises

1. Find $\phi(2.65)$, $\phi(-1.46)$, $\phi(0)$.
2. Find $t$ if $\phi(t) = .1257, .0325, .0034$, respectively.
3. Find the following areas under $\phi(t)$, and draw a figure in each case:
   (*a*) $\int_{0}^{\infty}, \int_{-1.2}^{1.2}, \int_{-\infty}^{1.2}, \int_{1.2}^{\infty}, \int_{-1.2}^{\infty}$.
   (*b*) $\int_{-.37}^{.37}, \int_{-.6745}^{.6745}$.
4. Find $t$, given the partial areas:

$$2\int_{0}^{t} = .5, \int_{0}^{t} = .27457, \int_{-t}^{t} = .999730.$$

5. Verify the percentages given in Figures 23 and 24.
6. (*a*) How far from the median of a normal distribution is the first quartile?
   (*b*) In a certain normal distribution $\bar{x} = 89$ and $Q_1 = 75.51$. What is $\sigma_x$?
7. For a normal distribution: $N = 1000$, $\bar{x} = 20$, $\sigma_x = 2$.
   (*a*) What is $E$?
   (*b*) Find the value of $Q_3$.
   (*c*) What values of $x$ will include the middle 500?
   (*d*) The middle 75%?
8. If $N = 300$, $\bar{x} = 75$, $\sigma_x = 15$, for a normal distribution:
   (*a*) What is the value of the first quartile?
   (*b*) The third quartile?
   (*c*) How many variates are between $x = 60$ and $x = 90$?
9. In a college the 8 grades A, A−; B, B−; C, C−; D, and F are given. On the assumption that mathematical ability is normally distributed, how many out of a total of 1000 should receive each grade? Assume that $\bar{x}$ is the boundary between the C and B− grades and that each grade interval is $.8\sigma$. What range in standard units on either side of $\bar{x}$ is thereby assumed to include all the grades?
10. What are the percentages of a normal distribution outside $\bar{x} \pm t\sigma$ for $t = 1, 2, 3$?

**7. Curve Fitting.** It should be remembered that a set of data collected and presented in the form of a frequency distribution is merely a sample of a general type called its universe. Other samples from that universe might yield somewhat different frequency distributions.

For certain purposes it may be desirable to fit a normal curve to a unimodal distribution which is reasonably symmetrical and appears to be of the normal type. The theoretical curve idealizes the recalcitrant observational data and smooths out the irregularities due to sampling fluctuations.

In fitting equation (3*a*) to a given distribution, we *assume* that

(*1*) *The given frequency N represented by a histogram equals the area under the curve, and*

(*2*) *The mean and standard deviation of the observed distribution equal, respectively, the mean and standard deviation of the theoretical distribution represented by the curve.*

A normal curve is a mathematical model of a hypothetical universe. In identifying such a universe with (3*a*) only its form is specified by the model. The parameters are (usually) unknown. An estimate of a parameter by the use of an appropriate function of the observed data is called *a statistic.* Assumption (2) above means,

then, that we replace each of the parameters by the corresponding statistic.[1]

The procedure of fitting a normal curve to an observed distribution will now be illustrated with the data of Table 21, p. 88. We substitute

$$\bar{x} = 47.712$$
$$\sigma_x = 5.772$$
$$N = 1000$$

in equation (3), and obtain

$$y = \frac{1000}{5.772\sqrt{2\pi}} e^{-\frac{(x-47.712)^2}{2(5.772)^2}}.$$

To make use of a table of standard ordinates in graphing this equation we transform it into standard units by setting

$$(a) \qquad t = \frac{x - 47.712}{5.772} = .17325x - 8.2661$$

and write

$$(b) \qquad y = \frac{N}{\sigma}\phi(t) = 173.25\phi(t).$$

Appropriate values to assign $x$ in equation $(a)$ are the *end-x* and *mid-x* values of the given distribution. The use of a computing machine in changing $x$ values into corresponding $t$ values is explained in §6, Chapter IV. Thus we obtain the values in the second column of Table 29. We may then enter the table in the Appendix for the corresponding ordinates, $\phi(t)$. These are converted into $y$ values by equation $(b)$. The curve may then be drawn by plotting

[1] It is shown in Part II that a better estimate of the variance in the universe is obtained by multiplying the variance of the observed distribution by $N/(N-1)$. Because of this fact some writers, denoting this result by $s^2$, *define* the variance of an observed distribution by

$$s^2 = \frac{1}{N-1}\sum_1^k f_i(x_i - \bar{x})^2.$$

The distinction between the two definitions is not an important one, in the author's opinion, for beginning students who are learning the descriptive methodology of statistics. And in curve fitting, the numerical difference is negligible because $N$ is fairly large. The distinction is important, however, in the theory of small samples (Part II).

TABLE 29. $t = .17325x - 8.2661$, $y = 173.25\phi(t)$

| $x$ | $t$ | $\phi(t)$ | $y$ | $f/c$ |
|---|---|---|---|---|
| 27.5 | −3.502 | .00086 | 0.15 | |
| 29.5 | −3.155 | .00275 | 0.48 | 0.25 |
| 31.5 | −2.809 | .00772 | 1.34 | |
| 33.5 | −2.462 | .01927 | 3.34 | 3.50 |
| 35.5 | −2.116 | .04253 | 7.37 | |
| 37.5 | −1.769 | .08344 | 14.46 | 14.00 |
| 39.5 | −1.423 | .14494 | 25.11 | |
| 41.5 | −1.076 | .22361 | 38.74 | 43.00 |
| 43.5 | −0.730 | .30563 | 52.95 | |
| 45.5 | −0.383 | .37072 | 64.23 | 61.25 |
| 47.5 | −0.037 | .39866 | 69.07 | |
| 49.5 | 0.310 | .38023 | 65.87 | 65.75 |
| 51.5 | 0.656 | .32230 | 55.84 | |
| 53.5 | 1.003 | .24124 | 41.79 | 39.00 |
| 55.5 | 1.349 | .16060 | 27.82 | |
| 57.5 | 1.696 | .09469 | 16.41 | 16.75 |
| 59.5 | 2.042 | .04960 | 8.59 | |
| 61.5 | 2.389 | .02299 | 3.98 | 5.75 |
| 63.5 | 2.735 | .00948 | 1.64 | |
| 65.5 | 3.082 | .00346 | 0.60 | 0.75 |
| 67.5 | 3.428 | .00111 | 0.19 | |

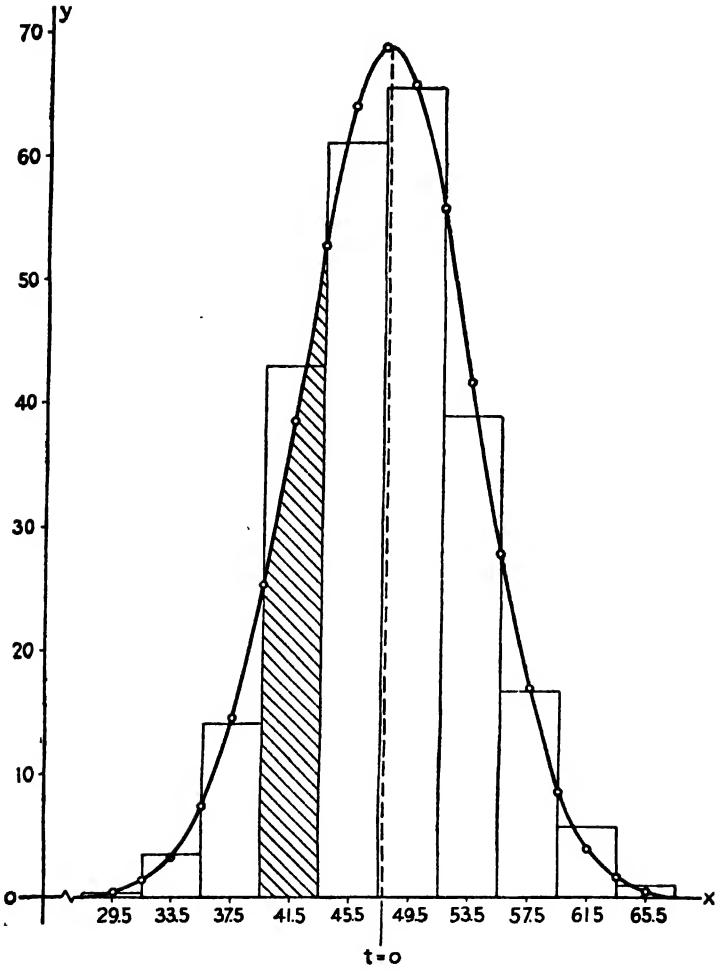
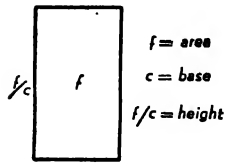



FIG. 25 — NORMAL CURVE FITTED TO HISTOGRAM REPRESENTING WEIGHT DISTRIBUTION OF GLASGOW SCHOOLGIRLS (TABLE 21)

The smooth curve is plotted from the points $(x, y)$ given in Table 29. The column headed $f/c$ in that table gives the heights of the rectangles in the histogram, $c = 4$. When both the curve and the histogram are to be drawn, it is best to draw the curve first so that the presence of the histogram will not prejudice one into trying to make the curve fit the histogram.

the $x$ and $y$ values. (Figure 25.) The curve should be drawn so as to be symmetrical with respect to the ordinate at the mean and its points of inflection should be at a distance from the mean equal to $\sigma$. The student should observe that every pair of $(x, y)$ values computed in Table 29 furnishes two points for the graph, each symmetrical to the other with respect to the mean ordinate. Both points should be used in drawing the curve but only the computed points should be left permanently in the graph.




After the curve is drawn, the histogram for the observed data may be constructed. The column headed $f/c$ gives the heights of the rectangles on the same scale as the ordinates of the curve.

**8. Graduation.** The areas under the fitted curve and over the class intervals are called theoretical frequencies. Thus in Figure 25 the shaded area represents the theoretical frequency corresponding to the observed frequency which is represented by the rectangle the mid-point of whose base is 41.5 pounds. The determination of the theoretical frequencies is called "graduation by the normal curve." It is a process of smoothing out the data to fit the curve. The method is shown in Table 30 for the data represented by Figure 25.

In order to enter a table of standard areas we must change the *end-x* values into $t$ values. These are given in the third column of Table 30. They are part of the values already computed for Table 29.

The entries in the column headed $A = \int_{-\infty}^{t}$ are the (*cum* $f$)/$N$ values of the standard curve for the given end-points. The entries in the column headed $\Delta A$ are obtained by differencing the preceding column. (See last paragraph of §9, Chapter I.) They are the percentages $p = f/N = \Delta A$ to be expected in the various intervals on the hypothesis of a normal distribution. Therefore $N\Delta A$ gives the *numbers* to be expected, that is, the theoretical frequencies.

The student should study this table until he becomes familiar with all the operations involved and what they mean. He should distinguish between the purposes of Tables 29 and 30.

**9. Purpose of a Graduation.** If, for the distribution of graduated frequencies, the mean, standard deviation, and total frequency are found, their values will be precisely those of the corresponding moments in the observed frequency distribution. This must be so, because these were the conditions imposed in the process of gradu-

TABLE 30

| *Observed Frequency* | *Boundary* $x$ | $t$ | $A = \int_{-\infty}^{t}$ | $\Delta A$ | $N\Delta A$ = *Theoretical Frequency* |
|---|---|---|---|---|---|
| | $-\infty$ | $-\infty$ | .0000 | | |
| 1 | | | | .0025 | 2.5 |
| | 31.5 | −2 809 | .0025 | | |
| 14 | | | | .0147 | 14.7 |
| | 35 5 | −2 116 | 0172 | | |
| 56 | | | | .0602 | 60.2 |
| | 39 5 | −1.423 | 0774 | | |
| 172 | | | | .1553 | 155 3 |
| | 43 5 | −0 730 | .2327 | | |
| 245 | | | | 2527 | 252.7 |
| | 47.5 | −0.037 | .4854 | | |
| 263 | | | | .2587 | 258.7 |
| | 51 5 | 0 656 | .7441 | | |
| 156 | | | | 1674 | 167.4 |
| | 55 5 | 1 350 | .9115 | | |
| 67 | | | | .0679 | 67.9 |
| | 59.5 | 2.042 | .9794 | | |
| 23 | | | | .0175 | 17.5 |
| | 63.5 | 2.735 | .9969 | | |
| 3 | | | | .0031 | 3.1 |
| | $\infty$ | $\infty$ | 1.000 | | |
| Totals | | | | 1.0000 | 1000.0 |

ation. Moreover, the observed values of skewness and kurtosis as given by $\alpha_3$ and $\alpha_4$ will not differ appreciably from the theoretical values if the fitting of the normal curve to the observed distribution was justified.

Since the above parameters characterize a distribution, the observing student may wonder why a distribution should be graduated if the values of these constants are unaltered in the process.

There are three main reasons why a student should be taught to graduate a curve. The first, and least important, has to do with the use of a smooth curve in place of a jagged sample. The second, and most important, is that it is necessary for the mathematical development of statistics that the mathematician should be told what assumptions he may make. These usually depend on the types of frequency curves which can be depended on to fit phenomena.... A third reason, intermediate in importance between the other two, is that in testing *a priori* theories in various fields, it is often necessary to test the efficacy of the frequency distributions which are results of these theories.[1]

The second and third of the above reasons may seem somewhat abstruse, but it is not easy to give completely satisfactory explanations of them at this level of exposition. About all we can say at this time is that the distribution of variation of a variable $x$ about its mean value is a fundamental statistical concept and in certain theoretical investigations it is very important that we have mathematical functions which are capable of representing such distributions. This is particularly true in sampling theory which will be discussed in Part II.

The first reason is more readily understood. Occasionally in practical problems it may be desirable to use the theoretical frequencies obtained by graduation in place of the observed data which probably contain irregularities due in part to grouping, in part to sampling fluctuations. We cite here two illustrations.

*Example* 1. A company which operates a chain of men's haberdashery stores planned to bring out a new line of about 100,000 light weight sport shirts suitable for camping, hunting, etc. The question arose as to the determination of the number of each size that should be ordered from the factory. Their previous distribution of sizes had not been satisfactory because the demand for certain sizes had been different from the number manufactured. Therefore the statistical department was requested to recommend the distribution of the proposed order according to neck sizes. The solution of the problem hinged upon the availability of data giving the measurements of neck circumferences of a large sample of men. Satisfactory data were found in the "Reports of the Medical Department of the United States Army in the World War," which gave a table of the

[1] *Journal of the American Statistical Assoc.*, vol. XXVI, March 1931, Supplement, p. 36.

neck measurements in centimeters of 95,102 white troops at demobilization. Since these data are tabulated in class intervals which are slightly different from the ranges used in standard shirt-band sizes, a slight adjustment was necessary. But essentially a normal curve was fitted to this distribution and the graduated frequencies were taken as the number of potential customers for each shirt size. The result was quite satisfactory.

*Example* 2. A well known and interesting illustration of the desirability of smoothing occurs in the census returns. The census takers' records show more persons alive at age 30 than at age 29, more at age 35 than at age 34, more at 40 than at 39, etc. This is probably due to the fact that men (as well as women) do not tell their exact ages. A person who is actually 41 or 42 and known to be 40 or so, says he is 40. The recorded data show artificial bumps at every age which is a multiple of 5. Naturally the Census Bureau prefers the smoothed results to be observed. The student should not infer that the curve used to smooth these data is the normal type. The "life curve" is a continuously decreasing function. However, the same kind of quinquennial irregularity occurs in other actuarial data which do approximate the form of a normal curve. Many examples are given in Elderton, *Frequency Curves and Correlation.*

**10. Probability.** A frequency curve is sometimes called a probability curve. The link connecting frequencies with probabilities has its starting point in the following definition:

DEFINITION. *If out of $N$ mutually exclusive and equally likely events, $f$ are distinguished by some property $A$, the probability of an event bearing the property $A$ is $f/N$.*

The definition implies that probability is measured by a number in the range 0 to 1, the lower limit denoting impossibility and the upper limit denoting certainty.

Since the total area under the curve represented by (4) is unity, any partial area denoted by (7) can be interpreted as the probability that a value of $t$ selected at random from a normal distribution (4) lies between $t = a$ and $t = b$.

*Example* 1. Refer to the data of Table 8, Chapter I. Let us assume that a normal curve was fitted to this distribution and that the fit seemed (by visual inspection) to be reasonably good. Generalizing on the experience shown in the table, the telephone company wishes to estimate the probability that a call (of the same type of message as that in the table) will be between (say) 500 seconds and 600 seconds in length.

*Solution.* Using (10),

$$a = (500 - 477.3)/148.5 = 0.15,$$
$$b = (600 - 477.3)/148.5 = 0.83.$$

Under the implicit assumptions, the required probability is $P = \int_{0.15}^{.83} = 0.24.$

*Example* 2. Referring to Example 1 above, find the probability that the length of a telephone call will differ numerically from the mean of the table by as much as 5 minutes.

*Solution.* We find $|t| = 300/148.5 = 2.02$. The probability of a deviation not greater, numerically, than 300 seconds is $P = 2\int_0^{2.02} = 0.96$, approximately. Then the probability of a numerical deviation as large as (or larger than) 300 seconds is $Q = 1 - P = 0.04$. This would be represented graphically by the area under the curve *outside* $t = \pm 2.02$.

**11. Probability Paper.** The cumulative frequencies for the normal $\phi(t)$ curve are given by $A = \int_{-\infty}^{t}$. As $t$ varies from $-\infty$ to $+\infty$, $A$ varies from 0 to 1, and for the finite range $t = \pm 3$ (commonly met in practice) $A$ varies from 0.00135 to 0.99865. (Verify.) Regarding $A$ as a function of $t$, values of $(t, A)$ from the tables may be plotted and the resulting points joined by a smooth curve.

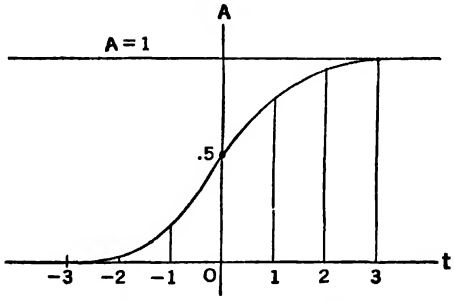



Fig. 26.—Ogive of the Normal Curve

When graphed on an algebraic scale this curve is the ogive of the normal curve. It is also called the integral curve of $\phi(t)$. As indicated in Figure 26, the ordinate of the ogive is zero at $t = -\infty$, .5 at $t = 0$, and the ogive approaches the line $A = 1$ asymptotically.

Now imagine the vertical scale of Figure 26 stretched in such a way that the ogive becomes a straight line. The stretching required will be greatest around the line $A = 0.5$ and gradually diminish as the distance from this line increases.

Paper so ruled that the $(t, A)$ graph is a straight line is called *probability paper*. It is readily obtainable[1] and is convenient for many purposes. Thus, by plotting *cum f* for an observed distribution on probability paper, one may observe how closely it approxi-

[1] The Codex Book Company, New York.

mates a straight line and hence get an idea of how nearly normal it is. One may thus locate graphically the median, quartiles, etc., and estimate frequencies between given limits.

A more complete discussion giving references to writers who suggested and developed the use of probability paper may be found in the *Journal of the American Statistical Association*, vol. XXVI, June 1931, p. 178.

## Exercises

**1.** Construct three normal curves on the same axes according to the following specifications. Compute ordinates at intervals of $.5\sigma$ from the mean in the range $\bar{x} \pm 3\sigma$.

| Curve | $\sigma_x$ | $\bar{x}$ | $N$ |
|---|---|---|---|
| $A$ | 10 | 50 | 400 |
| $B$ | 10 | 50 | 800 |
| $C$ | 10 | 50 | 1200 |

Suggested form for computations:

| $x$ | $t$ | $\phi(t)$ | $y$ | | |
|---|---|---|---|---|---|
| | | | $A$ | $B$ | $C$ |
| 20 | $-3$ | | | | |
| — | — | | | | |
| — | — | | | | |
| — | — | | | | |
| 80 | 3 | | | | |

**2.** Construct three normal curves on the same axes according to the following specifications. Compute ordinates at intervals of $.5\sigma$ from the mean.

| Curve | $\sigma_x$ | $\bar{x}$ | $N$ |
|---|---|---|---|
| $A$ | 15 | 50 | 1000 |
| $B$ | 10 | 50 | 1000 |
| $C$ | 5 | 50 | 1000 |

Suggestion:

| $x$ | | | $t$ | $\phi(t)$ | $y$ | | |
|---|---|---|---|---|---|---|---|
| $A$ | $B$ | $C$ | | | $A$ | $B$ | $C$ |
| 5 | 20 | 35 | $-3$ | | | | |
| — | — | — | — | | | | |
| — | — | — | — | | | | |
| — | — | — | — | | | | |

Observe that:

$$y_C = 200\phi(t)$$
$$y_B = \tfrac{1}{2}y_C = 100\phi(t)$$
$$y_A = \tfrac{1}{3}y_C = \frac{200}{3}\phi(t).$$

3. Verify the entries in Tables 29 and 30.
4. For the following distribution:
   (*a*) Find the equation of the best fitting normal curve, and plot the curve and histogram.
   (*b*) Find the graduated frequencies.

| *mid-x* | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| *f* | 1 | 4 | 6 | 4 | 1 |

5. Graduate the distribution in Table 8, §11, Chapter I. Also find the ordinates of the best fitting normal curve and plot the curve and histogram.
6. A distribution of the weekly wages of 906 anthracite miners showed the following results:

$$\bar{x} = \$36.13 \qquad \alpha_3 = 0.007$$
$$\sigma_x = \$8.87 \qquad \alpha_4 = 3.02$$

   Assuming a normal distribution, estimate the number of the 906 miners who received weekly wages (*a*) in excess of \$45, (*b*) less than \$25.
7. An urban electric railway company operating a large city subway uses thousands of electric light bulbs in its underground stations. On January 1, 1947, the company put into service 5000 new light bulbs. Let it be assumed that these 5000 bulbs will have a mean life of 50 days, a standard deviation of 19 days, and that their lives conform to the normal curve.

   If January 1 is counted as a full day in the life of the bulbs: (*a*) How many bulbs out of the 5000 new ones would have had to be replaced by midnight January 31, 1947? (*b*) How many by March 10, 1947?
8. Which properties of the normal curve may be used as criteria in passing judgment on the normality of an observed distribution? Would you say that the distributions referred to in Table 23 are approximately normal?
9. Graph the ogive of the normal curve by plotting values of $(t, A)$ in the range $t = \pm 3$, (*a*) on an algebraic scale, (*b*) on probability paper.
10. What famous mathematicians' names are associated with the normal curve? When did these men live? Which of them should most appropriately be credited with the discovery of this curve?
11. (*Camp*) The standard deviation of a certain set of 100,000 high school grades was 11%, and the mean grade was 78%. Assume the distribution to have been normal, and, being careful not to confuse percentage in the sense of grade with a percentage of frequency, answer the following questions: How many grades were (*a*) above 90%, (*b*) below 70%? (*c*) What

was the highest grade of the lowest 1000? (*d*) Within what limits did the middle 90,000 lie? (*e*) What was the semi-interquartile range?

**12.** (*Camp*) Answer all the questions of Exercise 11 with reference to a set of 100,000 grades in which the median was 83% and $Q_3$ was 90%. Also find $\sigma_x$.

**13.** In a certain normal distribution, $N = 1000$, $\bar{x} = 50$, $\sigma_x = 10$. For this distribution:

(*a*) Convert the following $x$'s into the corresponding $t$'s,

| $x$ | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | | | | | | | | | | | | | | | |

(*b*) Find from the tables the values of $\phi(t)$ for the $t$ values in (*a*).

(*c*) Convert the $\phi(t)$ values obtained in (*a*) into $y$ values.

(*d*) Plot the $(x, y)$ values in (*a*) and (*c*) and draw a smooth curve through them.

(*e*) Find the cumulative relative frequencies, $A = \int_{-\infty}^{t}$, for the values of $t$ in (*a*).

(*f*) Difference your results in (*e*) by finding $\Delta A$.

(*g*) Convert the percentages in (*f*) into frequencies.

(*h*) Explain the meaning of your results in (*g*) with reference to the figure for (*d*).

(*i*) Find the number of variates between $x = 42$ and $x = 74$.

(*j*) Find the values of $x$ for which $cum\ f = 250, 600, 750$, respectively.

**14.** Given a normal distribution in which $N = 800$, $\bar{x} = 40$, $\sigma_x = 7$. Find the numerical value of each of the following.

$$Q_1,\ Q_2,\ Q_3,\ E,\ N \int_{t=0}^{t=s}.$$

**15.** Suppose $N = 5000$ variates are normally distributed such that $\bar{x} = 50$ and $E = 13.49$. Without using the tables find the value of the following: quartiles, median, mode, standard deviation, mean deviation, $x$ for which $cum\ f = 1250$.

**16.** Suppose there are $N$ values of a variable $v$ which are normally distributed with mean = 0 and variance = 25.

(*a*) Give the equation of the curve which represents the distribution.

(*b*) If there are 793 values between $v = -5$ and $v = 0$, determine $N$.

(*c*) What percent of $N$ have values larger than $v = 10$?

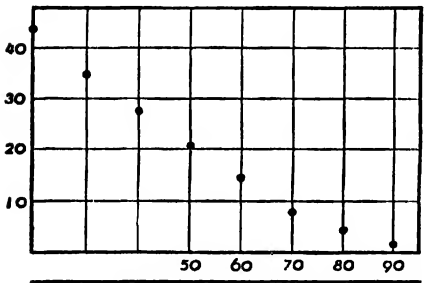
(*d*) Determine the value of $v$ for which $cum\ f = .75N$.

# CHAPTER VII

## CURVE FITTING

**1. Empirical Expressions.** The preceding chapters have dealt with the description and characterization of frequency distributions. We have considered three general methods of description: (1) graphical devices, (2) the method involving calculation of averages and measures of dispersion, (3) the method which is sometimes called *analytical.* This latter method consists in describing the distribution by an equation, and we considered only one such analytical expression, the normal curve.

*Example* 1. Expectation of Life[1] at various ages.




| Age | Expectation |
|---|---|
| 20 | 42.20 |
| 30 | 35.33 |
| 40 | 28.18 |
| 50 | 20.91 |
| 60 | 14.10 |
| 70 | 8.48 |
| 80 | 4.39 |
| 90 | 1.42 |

However, another branch of statistics is concerned with data which may not be classed under frequency distributions, but which may be described by simple equations.

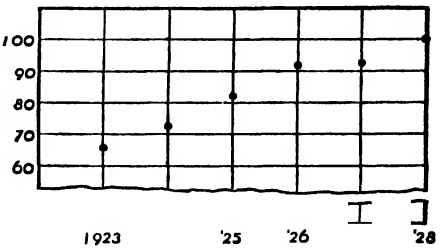
When one variable is a function of another in applied mathematics the mathematical relation between them is not always known. As we mentioned in Chapter II, the only information regarding this functional relationship may be a set of pairs of values obtained by experimental or observational means. These pairs of values may be regarded as coördinates of points and plotted. In doing so, the values of the variable which is regarded as independent are taken as abscissas, and those of the dependent variable as ordinates.

The general problem in such cases is to find, if possible, an analytic

[1] By expectation of life at any age is meant the average number of years lived by persons attaining that age, as given in the *American Experience Mortality Table.*

expression of the form $y = f(x)$ for the functional relationship suggested by the data. Equations obtained to fit observed data as well as possible are called empirical to distinguish them from the rational expressions of pure mathematics which can be derived from reasoning. This general problem is called *curve fitting*. It is also sometimes referred to as "smoothing" the given data.

*Example* 2. Yearly Production of Cigarettes in the United States




| Year | Billions |
|---|---|
| 1923 | 66.7 |
| 1924 | 72.7 |
| 1925 | 82.3 |
| 1926 | 92.1 |
| 1927 | 93.0 |
| 1928 | 100.0 |

We will consider three types of functions: *linear*, *quadratic*, and *exponential*.

**2. Linear Functions.** We know from algebra that the general form of a linear equation in two variables is

$$Ax + By = C$$

where $A$, $B$, and $C$ are arbitrary constants.

When $B \neq 0$, the equation may be solved for $y$, giving $y = -(A/B)x + C/B$ which is of the form

$$y = mx + k \tag{1}$$

and which is the form we will ordinarily use to represent a straight line.

The special cases where $A$ or $B$ or $C$ are zero is as follows:

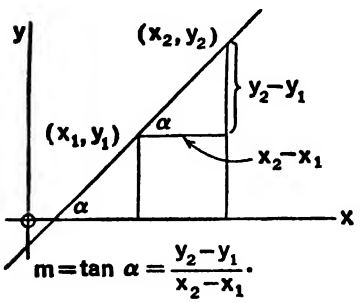
When $A = 0$, then $y = C/B$, which is of the form $y = k$. This is a line parallel to the $x$-axis. When $B = 0$, the equation takes the form $x = k$ which is a line parallel to the $y$-axis. When $C = 0$, then $Ax + By = 0$ which is a line passing through the origin.

The graph of (1) is a straight line (which explains the term "linear"). A characteristic property of a linear function is revealed at once by its graph. This is the fact that the ratio of a change in $y$ to the corresponding change in $x$ is *constant*. Thus, if two points $(x_1, y_1)$ and $(x_2, y_2)$ are chosen on the line, the value of the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

is independent of the points chosen. This ratio gives the average rate of change of any function over the interval $\Delta x = x_2 - x_1$. In the case of a *linear* function, $m$ defines the *rate of change* of the function.

Graphically, $m$ is the *slope* of the line. It is the tangent of the *angle of inclination* $\alpha$ (alpha) which the line makes with the positive $x$-axis.[1] Lines having the same slope are parallel, and conversely.




It is shown in analytic geometry that we may obtain the slope of a straight line from its equation if we solve for $y$ and take the coefficient of $x$. Thus in $2x - y = 5$, $y = 2x - 5$ and the slope is 2.

Conversely, if we know the slope of a line and the coördinates of any point on the line we can write its equation from the relation

$$y - y_1 = m(x - x_1) \tag{2}$$

which is called the *point-slope* form of a straight line. Thus, given that $(2, -1)$ is a point on a line whose slope is 2, the equation of the line is therefore $y + 1 = 2(x - 2)$ or $2x - y = 5$.

Or again, remembering that $m$ is defined by a ratio involving the coördinates of two points on a line, we can obtain the equation of a line if we know any two points which lie on it. From the definition of $m$ and (2), we have

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{3}$$

which is known as the *two-point* form of a straight line. Thus, given that $(2, -1)$ and $(6, 7)$ are two points on a line, its equation is

$$y + 1 = \frac{7 + 1}{6 - 2}(x - 2) \quad \text{or} \quad 2x - y = 5.$$

**3. Quadratic Function.** A quadratic function of a variable $v$ is a polynomial of the second degree in $v$ which may be expressed in the form $Av^2 + 2Bv + C$ where $A$, $B$, and $C$ are fixed real numbers.

[1] When the line is vertical, $\alpha = 90°$ and $m$ does not exist. Then $\Delta x = 0$ and division by zero is excluded in our algebra.

The minimum value of such a function is useful in statistics. We have

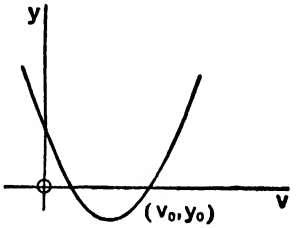
$$Av^2 + 2Bv + C = \frac{1}{A}[A^2v^2 + 2ABv + AC]$$

$$= \frac{1}{A}[(Av + B)^2 + (AC - B^2)].$$

Since $(Av + B)^2$ is positive or zero and $(AC - B^2)$ does not involve the variable, we have the following:

**Theorem I.** *If $A$ is positive the minimum value of $Av^2 + 2Bv + C$ occurs when $Av + B = 0$; the minimum value is $(AC - B^2)/A$.*

The graph of the equation $y = Av^2 + 2Bv + C$, $(A > 0)$, is a parabola which opens upward and whose vertex is where $v = -B/A$. Of course the function has its minimum value at this vertex, *viz.*: $(v_0, y_0)$ where $v_0 = -B/A$, $y_0 = (AC - B^2)/A$.




## Exercises

1. (*Wilson and Tracy*) The premium ($\$y$) on a \$1000 life insurance policy for various ages ($x$ years) is given in the following table. Draw a graph exhibiting $y$ as a function of $x$. Estimate from the graph the premium at age 32 and at age 43; also the age at which the premium is \$52.

| $x$ | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 18.78 | 21.02 | 23.86 | 27.54 | 32.36 | 38.83 | 47.68 | 59.88 | 76.94 |

2. Find an equation of each of the lines through two points given as follows: (*a*) (2, 6), (4, 5); (*b*) (0, 3), (1, 6).
3. Find the equation of a line through the point (2, 3) and parallel to the line $4x + 5y = 7$.
4. (*a*) Find the value of $x$ for which $f(x) = 2x^2 - 8x + 9$ has a minimum value. (*b*) What is this minimum value? (*c*) Draw a graph of $y = f(x)$ and show the meaning of your answers to (*a*) and (*b*).
5. How would the theorem in §3 be affected if $A < 0$?
6. Prove that the second moment of $x$ is a minimum when taken about the mean of $x$.

*Hints. Solution* 1.

Let
$$f(v) = \frac{1}{N}\sum_1^N (x_i - v)^2$$
$$= v^2 - 2\bar{x}v + \frac{1}{N}\sum_1^N x_i^2.$$

By the theorem of §3, show that $f(v)$ is a minimum when $v = \bar{x}$.

*Solution* 2. By definition,
$$\mu_2 = \frac{1}{N}\sum_1^N (x_i - \bar{x})^2$$
$$\nu_2 = \frac{1}{N}\sum_1^N (x_i - v)^2, v \neq \bar{x}.$$

Is $\mu_2 < \nu_2$?

*Solution* 3, for calculus students. From $f(v)$ as derived above,
$$f'(v) = -\frac{2}{N}\sum_1^N (x_i - v).$$

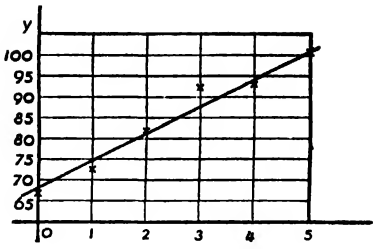
Set $f'(v) = 0$ and solve for $v$. Since $f''(v) > 0$, $v = \bar{x}$ yields a minimum, not a maximum.

7. Show that the value of $k$ for which $f(k) = Nk^2 + 2k(m\sum_1^N x_i - \sum_1^N y_i) + C$ is a minimum is defined by
$$m\sum_1^N x_i + Nk = \sum_1^N y_i.$$

**4. Fitting a Straight Line.** The preceding discussion is intended as a basis for the presentation of certain methods of fitting a line to data. The equation $y = mx + k$ represents a family or set of lines corresponding to different values of the arbitrary constants $m$ and $k$. As noted previously, such constants are called parameters. The process of finding the best fitting line for any given data consists in determining $m$ and $k$. By "best fitting" we mean best under a criterion of approximation specified by a method. We will consider three such methods: (*a*) *graphical*, (*b*) *the method of moments of ordinates*, (*c*) *the method of least squares.*

**5. Graphically.** A straight line is drawn (preferably with the aid of a transparent ruler) to fit as closely as possible the plotted points. To find the equation of this line, select two points on the line and estimate their coördinates $(x_1, y_1)$ and $(x_2, y_2)$. Substituting these coördinates in the "two-point" form of the line (3), we get the desired equation.

If the first point is chosen so that $x_1 = 0$ the numerical work of simplifying the equation is somewhat lessened.




| x | | y |
|---|---|---|
| (1923) | 0 | 66.7 |
| | 1 | 72.7 |
| | 2 | 82.3 |
| | 3 | 92.1 |
| | 4 | 93.0 |
| (1928) | 5 | 100.6 |

*Example* 3. Fit a line graphically to the data in Example 2.

We take the origin of $x$ at 1923, hence from the figure ($x_1 = 0$, $y_1 = 67$) and ($x_2 = 5$, $y_2 = 100$).

By equation (3),

$$y - 67 = \frac{100 - 67}{5} x$$

Therefore,

$$y = 6.6x + 67$$

is the required equation.

The graphical method is open to the objection that it depends upon the judgment of the investigator. Different people will locate the line in different positions and therefore obtain different equations. However, where only approximate results are needed it is usually quite satisfactory.

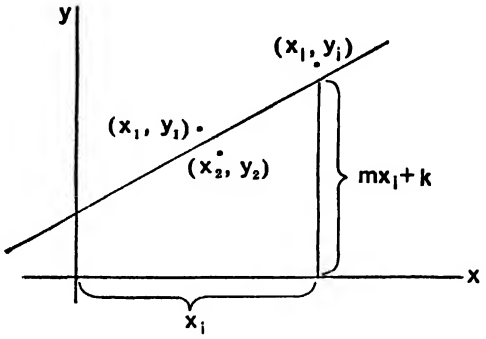
**6. Method of Moments.** In equation (1) $y$ is not only a function of $x$ but it is also a function of the *parameters* $m$ and $k$. This functional relationship may be expressed symbolically by the notation $f(x, m, k)$. Given the functional form of a curve $y = f(x, a, b, c, \cdots)$ the parameters $a, b, c, \cdots$ may be determined by obtaining expressions for as many moments of the *computed* or *functional* $y$'s as there are parameters in the function and equating these to the *numerical* moments of corresponding order of the observed or empirical $y$'s. A solution of the resulting equations, theoretically possible, gives the " best " values of the parameter. This is the *method of moments of ordinates*. For a set of $N$ values of $(x_i, y_i)$ the $r$th moment of $y$ is defined by the expression

$$\frac{1}{N} \sum_{1}^{N} x_i^r y_i$$

where $r$ is zero or a positive integer.

In fitting a straight line by this method we obtain two equations involving $m$ and $k$ if we equate the zero*th* and first moments of the observed $y$'s to the zero*th* and first moments, respectively, of the $y$'s

computed from the assumed equation $y = mx + k$. All moments are taken about the origin of $x$. These two equations may then be solved for $m$ and $k$. The procedure will be made clear by the figure and explanation below.




| x | $_0y$ |
|---|---|
| $x_1$ | $y_1$ |
| $x_2$ | $y_2$ |
| .. | .. |
| $x_i$ | $y_i$ |
| .. | .. |
| $x_n$ | $y_n$ |

| x | $_cy$ |
|---|---|
| $x_1$ | $mx_1 + k$ |
| $x_2$ | $mx_2 + k$ |
| .. | .. |
| $x_i$ | $mx_i + k$ |
| .. | .. |
| $x_n$ | $mx_n + k$ |

Suppose we are given $N$ pairs of values of $x$ and $y$. Denote the given or observed $y$'s by $_oy$ and the computed $y$'s by $_cy$. For the observed $y$'s, the first moment is $\frac{1}{N}\sum x_i y_i$, and the zero*th* moment is $\frac{1}{N}\sum x_i^0 y_i$. By a " computed $y$ " corresponding to any value of $x$ we mean the result obtained by substituting that value of $x$ in the equation $y = mx + k$, and solving for $y$. Thus, for any value of $x$, say $x_i$, we obtain $mx_i + k$ for the corresponding computed $y_i$. Graphically, it is an ordinate of the line. Therefore, the first moment of the computed $y$'s is $\frac{1}{N}\sum x_i(mx_i + k)$, and the zero*th* moment is

$\frac{1}{N}\sum x_i^0(mx_i + k)$. Applying the principle of moments we have

| | *observed* | *computed* |
|---|---|---|
| zero*th* moment | $\sum y_i$ | $= \sum(mx_i + k)$ |
| first moment | $\sum x_i y_i$ | $= \sum x_i(mx_i + k)$ |

where the summations run from 1 to $N$.

To solve for $m$ and $k$ we write the preceding equations in the following form:

$$(4) \qquad \begin{cases} m\sum x_i + kN = \sum y_i \\ m\sum x_i^2 + k\sum x_i = \sum x_i y_i. \end{cases}$$

By determinants,

$$(5) \qquad \begin{cases} m = \dfrac{\begin{vmatrix} \sum y & N \\ \sum xy & \sum x \end{vmatrix}}{\begin{vmatrix} \sum x & N \\ \sum x^2 & \sum x \end{vmatrix}} = \dfrac{(\sum y)(\sum x) - N\sum xy}{(\sum x)^2 - N\sum x^2} \\ k = \dfrac{\begin{vmatrix} \sum x & \sum y \\ \sum x^2 & \sum xy \end{vmatrix}}{D} = \dfrac{(\sum x)(\sum xy) - \sum y\sum x^2}{D}. \end{cases}$$

The determinant $D$ in the expression for $k$ is the same as that in the denominator of the expression for $m$. [In order to solve equations (4) for the values (5) it is assumed that $D$ does not vanish.] The terms in the expressions for $m$ and $k$ refer to the original data. When these expressions have been evaluated they replace $m$ and $k$ in the equation $y = mx + k$.

*Example* 4. Find by the method of moments the best fitting line for the data in Example 2.

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 0 | 66.7 | 0 | 0 |
| 1 | 72.7 | 72.7 | 1 |
| 2 | 82.3 | 164.6 | 4 |
| 3 | 92.1 | 276.3 | 9 |
| 4 | 93.0 | 372.0 | 16 |
| 5 | 100.6 | 503.0 | 25 |
| 15 | 507.4 | 1388.6 | 55 |

$$m = \frac{(507.4)(15) - 6(1388.6)}{(225) - 6(55)} = 6.86 \qquad k = \frac{15(1388.6) - 55(507.4)}{D} = 67.4.$$

Therefore,

$$y = 6.86x + 67.4.$$

**7. An Alternative Procedure.** In practice, it is sometimes easier to remember the procedure of fitting a line by the method of moments if one obtains the equations in (4) directly from the data instead of using the formulas for $m$ and $k$. This will involve the following three steps:

($a$) Substitute each of the given pairs of values in $y = mx + k$ and add the corresponding members of the resulting " equations." This gives the first equation in (4).

($b$) Multiply each " equation " in ($a$) by the coefficient of $m$ in that " equation " and add the corresponding members of the resulting " equations." This gives the second equation in (4).

($c$) Solve the equations simultaneously. This will give the required values of $m$ and $k$.

The algebraic statements which we designated " equations " (denoting that the statements are only approximately true) are called *observation equations* in the theory of errors. A linear combination of a set of linear observation equations is a true equation.

*Example.* Verify, for the data in Example 2, that the above procedure gives the same values of $m$ and $k$ as the formulas.

*Step* ($a$)

$$\begin{aligned} 66.7 &= 0m + k \\ 72.7 &= 1m + k \\ 82.3 &= 2m + k \\ 92.1 &= 3m + k \\ 93.1 &= 4m + k \\ 100.6 &= 5m + k \\ \hline 507.4 &= 15m + 6k \end{aligned}$$

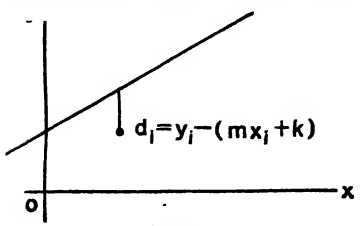
*Step* ($b$)

$$\begin{aligned} 72.7 &= m + k \\ 164.6 &= 4m + 2k \\ 276.3 &= 9m + 3k \\ 372.0 &= 16m + 4k \\ 503.0 &= 25m + 5k \\ \hline 1388.6 &= 55m + 15k \end{aligned}$$

*Step* ($c$)

Solving the equations, we obtain $m = 6.86$, $k = 67.4$, as before.

**8. Least Squares.** *Case I.* A standard method of fitting a curve to empirical data is one known as the method of *least squares*. Assume, as before, that the plotted data suggest the linear relationship $y = mx + k$. Let $d$ represent the difference between the ordinate of any given point and the corresponding ordinate of the line, that is, $d_i = [y_i - (mx_i + k)]$. These differences are called *residuals*. The method of least squares is based upon the following principle.




Case I

PRINCIPLE OF LEAST SQUARES. *The "best" estimate of a parameter is that for which the sum of weighted squares of the residuals is a minimum.*[1]

The sum is to be taken over all the observations that are subject to error. We shall assume that the observations are all of equal weight; consequently we may let each of the weights be unity. Then the parameters $m$ and $k$ are estimated by imposing the condition that $\sum_1^N d_i^2$ be a minimum. Now

$$\sum d^2 = \sum[y - (mx + k)]^2$$

$$= Nk^2 + 2mk\sum x + m^2\sum x^2 - 2k\sum y - 2m\sum xy + \sum y^2. \tag{6}$$

This is a quadratic polynomial in $k$. We may write it in the form

$$f(k) = Nk^2 + 2k(m\sum x - \sum y) + C \tag{6a}$$

where $C$ represents the terms not involving $k$. Then according to Theorem I the minimum value of $f(k)$ occurs when

$$k = \frac{\sum y - m\sum x}{N},$$

that is, when

$$Nk + m\sum x - \sum y = 0.$$

The right member of (6) is also a quadratic polynomial in $m$. We must choose $m$ so that

$$m\sum x^2 + k\sum x - \sum xy = 0.$$

These last two equations[2] are the same as (4). When obtained by the method of least squares they are called *normal equations*. Therefore the values of $m$ and $k$ in (5) determine the best fitting line by both the method of moments and of least squares. It can be shown that the two methods give the same result for any polynomial.[3]

It is interesting to observe that the sum of the residuals is zero. Thus it can easily be shown that $\sum[y - (mx + k)] = 0$, when the

[1] For further information about this principle and a discussion of weights, the following books are recommended: (*a*) Reference 4. (*b*) *Statistical Mathematics* — A. C. Aitken. Oliver and Boyd.

[2] The student of calculus would obtain these equations as follows. Let $f(m, k) = \sum(y - mx - k)^2$. Then differentiate $f(m, k)$ partially with respect to $m$ and $k$, respectively, and equate the results to zero.

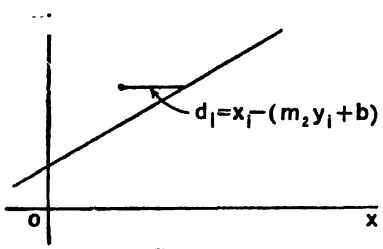
[3] See *American Mathematical Monthly*, September, 1923.

values given in (5) are substituted for $m$ and $k$. This property and the fact that the sum of the squares of the residuals is a minimum are quite analogous to two similar properties of the arithmetic mean, *viz.*,

(1) The sum of deviations from the mean is zero.

(2) The sum of the squares of deviations from the mean is less than the sum of the squares of such deviations taken from any other value, *i.e.*, $\mu_2 < \nu_2$.

*Case II.* In Case I distances between the points and the line were taken parallel to the $y$-axis. But we may just as logically, from a formal point of view, take distances parallel to the $x$-axis, and make the $x$ residuals the basis for a least squares criterion of best fit. Similarly, for the method of moments: we can set up two equations such that the first moment of the observed $x$'s equals the first moment of the computed $x$'s, and the zero*th* moment of the observed $x$'s equals the zero*th* moment of the computed. To do this let $x = m_2y + b$ represent the equation of the line. Then by the principle of moments we have



CASE II

$$\sum x = \sum (m_2 y + b)$$
$$\sum xy = \sum y(m_2 y + b).$$

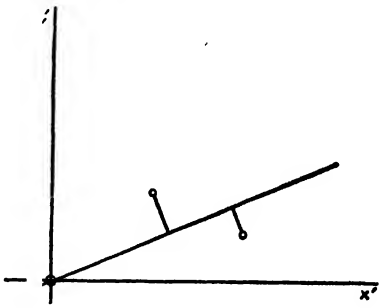
Solving for $m_2$ and $b$ we obtain

$$(7) \qquad \begin{cases} m_2 = \dfrac{\sum x \sum y - N \sum xy}{D} \\ b = \dfrac{\sum y \sum xy - \sum x \sum y^2}{D} \\ D = (\sum y)^2 - N \sum y^2. \end{cases}$$

If we determined $m_2$ and $b$ by making the sum of the squares of the $x$ residuals a minimum we would get the results given in (7). The expressions in (7) are those of (5) with $x$ and $y$ interchanged.

In general, Cases I and II will give different lines. Case I assumes that the observed points fail to fall on the line because of errors in the ordinates only. Case II assumes that only the $x$-coördinates are in error. In the application of curve fitting to economic data, etc., the formal mathematical procedure should not be used without

first verifying that the underlying assumptions involved in the procedure are justified. Inasmuch as the independent variable $x$ can be controlled in experimental and observational data, the errors usually exist only in the $y$'s. Therefore, in speaking of the best line by the method of moments or least squares it is conventional to mean the line which fits best in the sense of (5) rather than (7).



CASE III

*Case III (for calculus students).* A third line can be obtained which fits best in the sense that the sum of the squares of the perpendicular distances from the points to the line is a minimum.

Let us suppose the equation of this line to be in the form

$$y' = mx' + k$$

where $x' = x - \bar{x}$, $y' = y - \bar{y}$, and $(\bar{x}, \bar{y})$ is the mean of the observed data. The distance $d_i$ from this line to a point $(x_i', y_i')$ representing a pair of observed values (referred to their respective means as origin) is, from analytics,

$$d_i = \frac{y_i' - mx_i' - k}{\sqrt{m^2 + 1}}.$$

We wish to make $\frac{1}{N}\sum_1^N d_i^2$ a minimum. Therefore we are to choose $m$ and $k$ so that the function

$$f(m, k) = \frac{1}{m^2 + 1}\left\{\frac{1}{N}\sum_1^N (y_i' - mx_i' - k)^2\right\}$$

is a minimum. This function may be written in the form

$$f(m, k) = \frac{1}{m^2 + 1}(\sigma_y^2 + k^2 + m^2\sigma_x^2 - 2mr\sigma_y\sigma_x)$$

where $r$ is a convenient symbol defined by the relation

$$r\sigma_y\sigma_x = \frac{1}{N}\sum_1^N x_i'y_i'.$$

To make $f(m, k)$ a minimum we first put $k^2 = 0$. Then we equate to zero the first derivative with respect to $m$ and obtain

$$m^2 r\sigma_y\sigma_x - m(\sigma_y^2 - \sigma_x^2) - r\sigma_y\sigma_x = 0.$$

Solving for $m$ we have

$$m = \frac{(\sigma_y^2 - \sigma_x^2) \pm [(\sigma_y^2 - \sigma_x^2)^2 + 4r^2\sigma_y^2\sigma_x^2]^{1/2}}{2r\sigma_y\sigma_x}$$

Therefore the required equation is $y' = mx'$. Referred to the origin of $x$ and $y$, this is

$$y - \bar{y} = m(x - \bar{x})$$

where $m$ is determined above.

This line is the appropriate one to fit if there are errors in both $x$ and $y$ of the empirical data.

*A special problem under Case I.* Sometimes problems arise where the line to be fitted is restricted in some way. For example, the nature of the problem may require that the line shall pass through the origin. If this condition is imposed, (1) takes the form

$$y = mx.$$

The least squares estimate of the slope of this line depends upon various assumptions about the errors. If $y$ is subject to error and $x$ is free of error, and if the observations are all of equal weight, it is easy to show that

$$m = \frac{\sum xy}{\sum x^2}$$

by the principle of least squares. This principle will give different estimates of $m$ under different assumptions about the weights of the observations. Several particular solutions of the more general problem and some applications will be found in §15 of reference 4 on page 6. (See also our Exercise 11, p. 189.)

### Exercises

1. Fit a line to the following data by Case I:

*Ans.* $y = -.5x + 8$.

| $x$ | 6 | 7 | 7 | 8 | 8 | 8 | 9 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 5 | 5 | 4 | 5 | 4 | 3 | 4 | 3 | 3 |

2. Show that $\sum d = 0$ for Exercise 1.
3. Using the values given in (5) for $m$ and $k$ show that $\sum[y - (mx + k)] = 0$.
4. Verify the expressions for $m_2$ and $b$ given in (7). How would you modify the "alternate procedure" so it will apply to $m_2$ and $b$?
5. Fit a line to the data of Example 2 by the method of Case II.
6. Show that the formulas in (5) fail when the $x$'s are all equal. *Hint.* Replace $x$ by a constant $c$ in the denominator $D$.

**9. Simplification.** The formulas for $m$ and $k$ may be simplified. For certain purposes it may be desirable to make the transformations $x' = x - \bar{x}$ and $y' = y - \bar{y}$. This has the effect, graphically, of

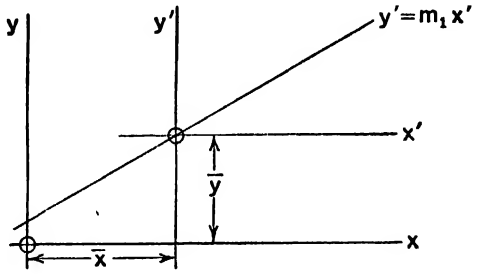



FIG. 27

translating the origin to the point $(\bar{x}, \bar{y})$ so that the $y$-axis is moved to the value $\bar{x}$, and the $x$-axis is moved to the value $\bar{y}$. Let the equation of the line with reference to these new axes be $y' = m_1x' + k_1$. The formulas for $m_1$ and $k_1$ will be the same as for $m$ and $k$ except that $x$ will be replaced by $x'$ and $y$ by $y'$. Hence

$$m_1 = \frac{N\sum x'y' - \sum x' \sum y'}{N\sum x'^2 - (\sum x')^2}$$

$$k_1 = \frac{\sum x'^2 \sum y' - \sum x' \sum x'y'}{N\sum x'^2 - (\sum x')^2}.$$

But since $x'$ is a deviation from the mean of $x$, $\sum x' = 0$. Similarly, $\sum y' = 0$. Hence the values of $m_1$ and $k_1$ reduce to

$$m_1 = \frac{\sum x'y'}{\sum x'^2}, \qquad k_1 = 0. \tag{8}$$

Therefore the line goes through the new origin, and its equation is

$$y' = m_1x' \tag{9}$$

where $m_1$ is defined in (8).

The above transformation may not lighten the computations unless the values of $x$ or $y$ are equispaced. However, it does simplify the theory in certain applications, particularly in correlation theory (Chapter VIII).

**10. Time Series.** If one of the variables is time, as in Examples 1 and 2, the data are called a time series. The best fitting line is then commonly called a trend line or trend. In the process of fitting a trend line, a first simplification, obviously, is to take the origin at one of the given dates as we did in Example 3. But a much greater simplification is possible, if the $x$'s are equispaced, as they usually are in a time series. Denote the common differences of the $x$'s by $c$ and the mid-date by $\bar{x}$. Then we may shift the origin to $\bar{x}$ and change the unit of measurement along the horizontal axis to $c$. Thus we may let

$$t = \frac{x - \bar{x}}{c} \tag{10}$$

where

$$\bar{x} = \frac{x_1 + x_N}{2} \tag{11}$$

if the $x$'s are equispaced.

Let us think now of our line in $(t, y)$ coördinates, and let its equation be $y = at + b$. Our problem is to find $a$ and $b$ numerically from the given data, as we found $m$ and $k$ before. Our normal equations will be

$$\begin{aligned} \sum y &= \sum (at + b) \\ \sum ty &= \sum (at + b)t. \end{aligned}$$

Since $\sum t = \frac{1}{c} \sum (x - \bar{x}) = 0$, and $\sum b = Nb$, the above equations are readily solved, giving

$$a = \frac{\sum ty}{\sum t^2}, \qquad b = \frac{1}{N} \sum y. \tag{12}$$

The student should remember that this simplification can be used only when the $x$'s are equispaced.

*Example* 5. Find the trend line for the following data. Here $c = 5$, and from (11) $\bar{x} = 10$.

| $x$ | $y$ | $t$ | $ty$ | $t^2$ |
|---|---|---|---|---|
| 0 | 12 | −2 | −24 | 4 |
| 5 | 15 | −1 | −15 | 1 |
| 10 | 17 | 0 | 0 | 0 |
| 15 | 22 | 1 | 22 | 1 |
| 20 | 24 | 2 | 48 | 4 |
| Sums | 90 | | 31 | 10 |

From (12),

$$a = \frac{31}{10} = 3.1, \qquad b = \frac{90}{5} = 18.$$

So the required equation is $y = 3.1t + 18$, with reference to the new origin and units. If we wish it in terms of $x$, we substitute

$$t = \frac{x - 10}{5}$$

and obtain $y = .62x + 11.8$.

*Example* 6. Same as Example 5, with another observation added. Note that when there is an even number of observations, the values of $t$ are fractional. In this case it is convenient to use the column headings $2ty$ instead of $ty$, and $4t^2$ instead of $t^2$.

| $x$ | $y$ | $t$ | $2ty$ | $4t^2$ |
|---|---|---|---|---|
| 0 | 12 | −5/2 | −60 | 25 |
| 5 | 15 | −3/2 | −45 | 9 |
| 10 | 17 | −1/2 | −17 | 1 |
| 15 | 22 | 1/2 | 22 | 1 |
| 20 | 24 | 3/2 | 72 | 9 |
| 25 | 30 | 5/2 | 150 | 25 |
| Sums | 120 | | 122 | 70 |

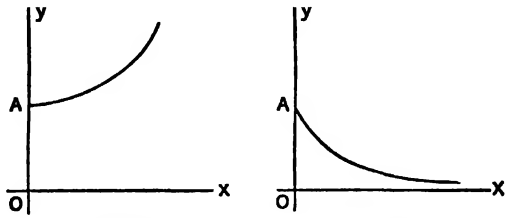
$$\bar{x} = 12.5, \qquad \sum ty = 61, \qquad \sum t^2 = 17.5$$
$$a = 3.49, \qquad b = 20$$
$$y = 3.49t + 20$$
$$y = 3.49\left(\frac{x - 12.5}{5}\right) + 20$$
$$y = .7x + 11.28.$$

**11. Exponential Trends.** When the given $y$ values form a geometric progression while the corresponding $x$ values form an arithmetic progression, the relationship between the variables is given by an exponential function, and the best fitting curve is said to describe an exponential trend. Data from the fields of biology, banking, and economics frequently exhibit such a trend. Thus the growth of bacteria is exponential. Money accumulating at compound interest follows the same kind of law of growth. And in business, sales or earnings may grow exponentially over a short period. Another familiar example is the increase in friction as a rope is coiled around a post. As the number of coils increases in arithmetic progression, the friction increases in geometric progression.[1] This explains why a few turns of the hawsers around the bitts at the wharf is sufficient to hold a large ship.

The characteristic property of this law is that the rate of growth, that is, the rate of change of $y$ with respect to $x$, at any value of $x$ is proportional to the value of the function for that value of $x$. The function

$$y = Ae^{Bx} \tag{13}$$

has this property.[2] The letter $e$ is a fixed constant, whereas $A$ and $B$ are parameters to be determined from the data. If $y$ decreases as $x$ increases, $B$ is negative. An interesting example of this case is the disappearance of radioactive substances like radium.




*When $B > 0$.* *When $B < 0$.*

FIG. 28 — GENERAL APPEARANCE OF THE GRAPH OF (13) FOR $x \geqq 0$ AND $A > 0$.

To assume that the apparent law of growth will continue is usually unwarranted, so only short range predictions can be made with any considerable degree of reliability. When the exponential character

[1] *Elementary Mathematical Analysis* — C. S. Slichter. McGraw-Hill.

[2] The student of calculus will understand that "rate of change" is used here in the derivative sense. For (13), $dy/dx = ky$.

of the observed phenomenon ceases a saturation point is said to be reached.

*The parameters A and B.* If we transform (13) so that it is linear with respect to its parameters we may use the methods for fitting a straight line to determine $A$ and $B$. To this end we first take the logarithms (to base 10) of both sides of (13), obtaining

$$\log y = \log A + (B \log e)x \tag{14}$$

which is of the form

$$Y = k + mx \tag{15}$$

where $Y = \log y$, $k = \log A$, $m = B \log e$.

If we look up the logarithms of the given $y$'s and denote them by $Y$, we may fit the equation $Y = mx + k$ to the $(x, Y)$ values by determining $m$ and $k$ by means of the formulas given in (5). In using these formulas we must remember to replace $y$ by $Y$. After $m$ and $k$ are determined, $A$ and $B$ may be obtained from the relations

$$A = \text{anti-log of } k$$
$$B = m/\log e, \text{ where } \log e = \log 2.718$$
$$= .4343.$$

The student may be interested to verify that the relation $Y = mx + k$ can be put back into the form (13). We may write (14) in the form

$$y = 10^{\log A + (B \log e)x}$$
$$= \{10^{\log A}\}\{10^{\log e}\}^{Bx}$$
$$= Ae^{Bx}.$$

The last step follows because $10^{\log_{10} N} = N$ by definition of logarithm.

*Example* 7. Find the exponential trend for the following data, and draw the curve.

| $x$ | $y$ | $Y$ | $xY$ | $x^2$ |
|---|---|---|---|---|
| 1 | 1.6 | .2041 | .2041 | 1 |
| 2 | 4.5 | .6532 | 1.3064 | 4 |
| 3 | 13.8 | 1.1399 | 3.4197 | 9 |
| 4 | 40.2 | 1.6042 | 6.4168 | 16 |
| 5 | 125.0 | 2.0969 | 10.4845 | 25 |
| 15 | | 5.6983 | 21.8315 | 55 |

From (5) we have,

$$D = (\sum x)^2 - N\sum x^2$$

$$m = \frac{1}{D}[\sum Y \sum x - N\sum xY]$$

$$k = \frac{1}{D}[\sum x \sum xY - \sum Y \sum x^2].$$

Therefore,

$$D = [(15)^2 - 5(55)] = -50$$

$$m = \frac{1}{D}[(5.6983)(15) - 5(21.8315)] = .4737$$

$$k = \frac{1}{D}[15(21.8315) - (5.6983)(55)]$$

$$= -.2813 = 9.7187 - 10.$$

And

$$\log A = 9.7187 - 10, \text{ hence } A = .5232$$

$$B = \frac{m}{.4343} = 1.091.$$

Therefore the required equation is

$$y = .5232e^{1.091x}.$$

When the $x$'s are equispaced, as here, the work may be simplified by using (10) and fitting a line

$$Y = at + b.$$

The problem now is essentially the same[1] as in §10 where $a$ and $b$ are defined in (12) except that we are now dealing with $(t, Y)$ coördinates instead of $(t, y)$.

The method is illustrated below.

| $t$ | $Y$ | $tY$ | $t^2$ |
|---|---|---|---|
| −2 | .2041 | −.4082 | 4 |
| −1 | .6532 | −.6532 | 1 |
| 0 | 1.1399 | 0.0000 | 0 |
| 1 | 1.6042 | 1.6042 | 1 |
| 2 | 2.0969 | 4.1938 | 4 |
| $t = x - 3$ | 5.6983 | 4.7366 | 10 |

[1] The critical reader will realize that fitting a straight line to the values of log $y$ is not quite the same as fitting an exponential to the values of $y$. However, the discrepancy usually does not affect the fit seriously. For a method which is free from this difficulty, see *Glover's Tables*, p. 468.

From (12)

$$a = \frac{\sum tY}{\sum t^2} = \frac{4.7366}{10} = .4737$$

$$b = \frac{1}{N}\sum Y = \frac{5.6983}{5} = 1.1397.$$

So

$$Y = 4737t + 1.1397.$$

Transforming this into $(x, Y)$ coördinates we have

$$Y = .4737(x - 3) + 1.1397$$
$$= .4737x - .2814$$

as before.

For purposes of plotting, predicting, or interpolating, values of $y$ in (13) may be obtained by means of the intermediate form (15). So, to sketch the curve

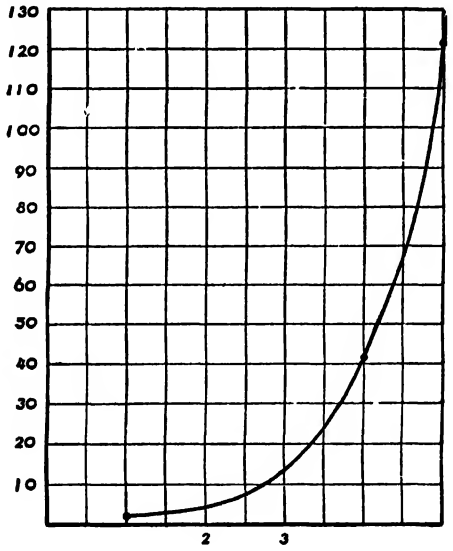



FIG 29

for this example, we first assign values to $x$ in the last equation, compute the corresponding values of $Y$, and then obtain the values of $y$ from a table of logarithms. These values are given in the following table. The curve in Figure 29 is sketched from the $(x, y)$ values in this table.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Y$ | 0.1923 | 0.6660 | 1.1397 | 1.6134 | 2.0871 | 2.5608 |
| $y$ | 1.56 | 4.63 | 13.79 | 41.06 | 122.2 | 363.8 |

**12. Further Remarks on the Exponential Function.** Equation (13) is sometimes called the compound interest law because it describes the way money would grow if interest were compounded continuously. If $P$ dollars are invested at a nominal rate $j\%$ compounded $m$ times a year, the amount $S$ after $x$ years is given by the formula

$$S = P\left(1 + \frac{j}{m}\right)^{mx}.$$

If $j$ is compounded continuously or, in other words, if $m$ is taken indefinitely large (written $m \to \infty$), the amount $S$ does not increase indefinitely but approaches a limiting value. We may write the expression for $S$ in the form

$$S = P\left[\left(1 + \frac{j}{m}\right)^{m/j}\right]^{jx}.$$

If we let $N = m/j$, we have

$$S = P\left[\left(1 + \frac{1}{N}\right)^{N}\right]^{jx}.$$

It can be shown in the calculus[1] that, as $N \to \infty$, the quantity $\left(1 + \frac{1}{N}\right)^{N}$ approaches the limit called $e$. Thus we have

$$\lim_{N\to\infty}\left(1 + \frac{1}{N}\right)^{N} = e = 2.718$$

This limit is also the base of the Napierian, or natural, system of logarithms. As $m \to \infty$ so does $N \to \infty$. Therefore in the ideal case of continuous conversion of interest, we have the limiting form

$$S = \lim_{m\to\infty} P\left[\left(1 + \frac{j}{m}\right)^{m/j}\right]^{jx}$$

$$= \lim_{N\to\infty} P\left[\left(1 + \frac{1}{N}\right)^{N}\right]^{jx},$$

that is

$$S = Pe^{jx}$$

which is of the form (13).

There are several other forms of the exponential function. For example, if we let $r = e^{B}$, (13) becomes

$$y = Ar^{x}$$

[1] The teacher can give appropriate references.

which is the general term of a geometric progression whose first term is $A$ and common ratio is $r$.

If $B$ is negative in $r = e^B$ then $r < 1$. So (13) is a decreasing function when $B$ is negative.

If we let $10^k = e^B$, (13) becomes

$$y = A10^{kx}.$$

Then $k = B \log_{10} e$ and $k$ differs from $B$ by the factor $\log_{10} e$. This factor is known as the *modulus* of the system of logarithms of base 10 with respect to the system of base $e$.

The value of the reciprocal of the modulus

$$\frac{1}{\log_{10} e} = 2.3025851 \cdots$$

is often useful. For example, suppose that the logarithm to base $e$ is required for a given number $N$ and tables to base 10 only are available. Let $\log_e N = x$. Then $e^x = N$, and $x \log_{10} e = \log_{10} N$, whence $x = \log_{10} N / \log_{10} e = 2.303 \log_{10} N$. (Hereafter, the base 10 will be understood unless otherwise indicated.)

**13. Ratio Charts.** In the graphical representation of data that exhibit an exponential trend, it is often desirable to use semi-logarithmic paper. Such paper has a logarithmic scale in the vertical direction and a uniform scale in the horizontal direction. (Figure 30.) A logarithmic scale is one in which the distance from $y = 1$ to $y = N$ equals $\log N$. A "cycle" of rulings spaced according to the logarithms of the integers from 1 to 10 is the unit of the vertical $\log y$ scale.

"Semi-log" paper may be constructed or purchased having one or more cycles. The appropriate number of cycles is determined by the range of $y$ values in the data to be plotted. If the bottom line of the first cycle is labeled 1 and taken as the origin of $\log y$ ($\log 1 = 0$), the beginning of the next cycle is read 10 ($\log 10 = 1$), the next one above that is read 100 ($\log 100 = 2$), etc. However, the beginning of the first cycle may be labeled with any number which is an integral power (positive or negative) of 10, as .01, .1, 10, 100, etc. Corresponding lines in successive cycles are labeled with numbers which are 10 times those in the preceding cycle. Since $y$ has no real logarithm if $y \leqq 0$, neither zero nor negative numbers are found on a logarithmic scale. Plotting a point whose semi-logarithmic co-

ordinates are $(x, y)$ is equivalent to plotting the point whose rectangular coördinates are $(x, \log y)$.

*Example* 8. Plot $y = 8\ (2^x)$ on semi-log paper.
*Solution.* Assigning values to $x$ we form the following table,

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 |

from which we obtain the semi-logarithmic graph shown in Figure 30.

We now state the following theorem.

**Theorem** II. *If $A$ is a positive constant, the $(x, \log y)$-graph of $y = Ae^{Bx}$ is a straight line.*

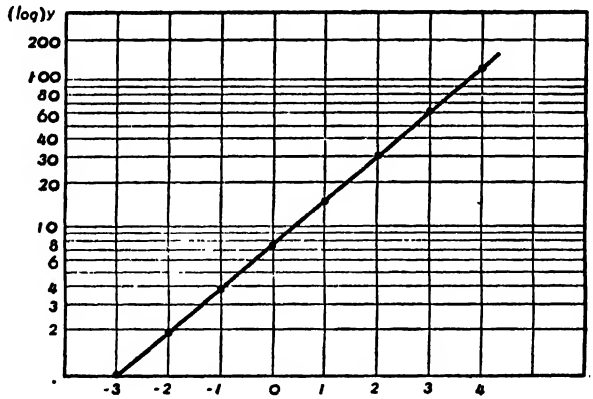



FIG. 30

*Proof:* Since (15) is linear in $x$ and $Y$, its graph in $(x, Y)$ rectangular coördinates is a straight line.

Semi-logarithmic graphs are also called *ratio charts.* Their usefulness depends upon the property of logarithms that

$$\log \frac{M}{N} = \log M - \log N.$$

It follows that the distance between any two ordinates of the chart measures the ratio between the values represented by these ordinates.

Thus if

$$\frac{y_1}{y_2} = \frac{y_3}{y_4}$$

then

$$\log y_1 - \log y_2 = \log y_3 - \log y_4$$

or

$$Y_1 - Y_2 = Y_3 - Y_4,$$

that is, equal ratios are represented by equal vertical distances. Likewise, if

$$\frac{y_1}{y_2} > \frac{y_3}{y_4}$$

then

$$Y_1 - Y_2 > Y_3 - Y_4$$

and the larger ratio is represented graphically by the larger distance. These differences of elevation are independent of any base line. The same percentage increase in $y$ is represented by the same addition to the height of $Y$ in all parts of the chart. Hence, it is easier to depict and discover percentage changes on ratio charts than on ordinary charts.

The analysis of time series in economic statistics is often facilitated by forming " link relatives " which are ratios of each ordinate (after the first) to the preceding ordinate. Thus, if $y_1, y_2, \cdots, y_n$ are the given values, the link relatives are

$$R_1 = \frac{y_2}{y_1}, \quad R_2 = \frac{y_3}{y_2}, \cdots, \quad R_{n-1} = \frac{y_n}{y_{n-1}}.$$

Any link relative $R$ denotes the percentage change in $y$ from one month (say) to the next. If the $y$'s are plotted on ratio paper they will lie on a straight line when the $R$'s are equal, on a curve bending upward when the $R$'s are increasing, and on a curve bending downward when the $R$'s are decreasing. It follows that if two curves are parallel on ratio paper their rate of increase (or decrease) is the same.

For further discussion of ratio charts the student is referred to the books of Bivins and Haskell (see §7, Introduction).

*Graphical determination of exponential function.* It follows from Theorem II that data giving a straight line when plotted on semi-logarithmic paper (with $x$ on the uniform and $y$ on the logarithmic scale) satisfy an equation of the form (13). Suppose that the

(straight line) graph has been drawn and one desires the exponential function which the line represents and the data satisfy. The constants $A$ and $B$ in (13) can be approximated by the following method.[1] We first observe that the slope of the line represented by (15) is given by

$$m = B \log e = \frac{Y_2 - Y_1}{x_2 - x_1}.$$

To determine the numerical value of $B$, take one cycle of $y$ (over which the graph extends) from any starting point and read the corresponding values of $x$ (Figure 31$a$), so that

$$B = \frac{Y_2 \quad Y_1}{(x_2 - x_1) \log e} = \frac{\log (y_2/y_1)}{\Delta x \log e} = \frac{\log 10}{\Delta x \log e} = \frac{2.303}{\Delta x}.$$

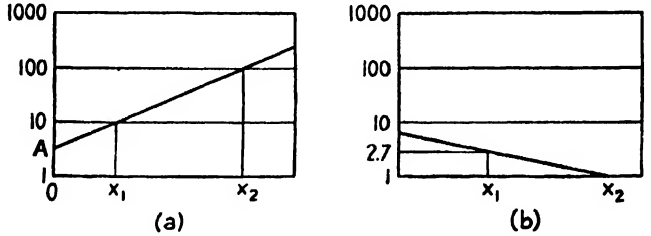



FIG. 31

In case the graph does not extend over one cycle, determine $x$ for $y = e$ and $y = 1$; then (Figure 31$b$)

$$B = \frac{\log e}{\Delta x \log e} = \frac{1}{\Delta x}.$$

The sign of $B$ is of course positive if the graph has a positive slope in the ordinary sense and is negative for a negative slope.

If the graph intersects the line $x = 0$, the value of $A$ can be read off at this intersection. If, in the data involved, the graph does not intersect the line $x = 0$, $A$ can usually be determined by finding $y$ for some convenient values of $x$ such as $Bx =$ some integer $n$, whereupon $A = y/e^n$ from equation (13).

In practical problems, the plotted points representing the data

[1] *Note on Semi-Logarithmic Graphs* — W. T. Lenser, The American Mathematical Monthly, vol. 49 (1942), pp. 611–613.

will not usually fall exactly on a straight line. But if they exhibit a linear trend one may draw (with the aid of a transparent ruler) the line that seems to fit them best. Then proceed as above.

*Example* 9. The uniform scale along the horizontal axis of a sheet of semi-logarithmic paper ranges from 0 to 10; along the vertical axis the logarithmic scale ranges from 100 to 1000. A straight line is drawn on the paper from the upper endpoint of the vertical scale to the midpoint of the horizontal scale. Determine (i) the equation of the exponential function represented by the line, (ii) the equation of the line in $(x, Y)$ coördinates.

*Solution* 1, using above method. $A = 1000$. $B = -1/(5 \log e) = (-2.3)/5 = -0.46$. Hence, the desired equation (i) is

$$y = 1000e^{-0.46x}.$$

The slope of the line is $m = B \log e = -\frac{1}{5}$ and its equation (ii) is

$$Y = 3 - 0.2x.$$

*Solution* 2. The line goes through the points (0, 1000) and (5, 100). Substitution of the first pair of coördinates into (13) gives $A = 1000$. Substitution of the second pair into $y = 1000e^{Bx}$ gives $100 = 1000e^{5B}$. Then $e^{-5B} = 10$ and $-5B = \log_e 10 = 2.303$, whence $B = -0.46$.

**14. Logarithmic Coördinate Paper.** A function of the form

$$y = kx^m \tag{16}$$

is called a *power function.* If $k > 0$ we have

$$Y = K + mX \tag{17}$$

where the capital letters denote the logarithms of the corresponding lower-case letters. Form (17) suggests the usefulness of *logarithmic coördinate* paper on which the rulings in both directions are at distances from the origin that are proportional to the logarithms of the numbers represented. To mark on this paper a point whose ordinary coördinates are $(X_1, Y_1)$ we plot the point whose rulings correspond to the numbers $x_1$ and $y_1$.

It is evident from (17) that the graph of (16) is a straight line on logarithmic coördinate paper. It also follows from (17) that the problem of fitting a curve of the form (16) to a set of observations can be reduced to the problem of fitting a straight line.

*Example* 10. A straight line is drawn on logarithmic coördinate paper through the points (4, 16) and (6, 54). Determine the function $y = f(x)$ which has that line as its graph.

*Solution* 1. Substitution of the coördinates of the given points into (16) gives

$$\begin{cases} 16 = k(4^m) \\ 54 = k(6^m). \end{cases}$$

Upon dividing each member of the first equation by the corresponding member of the second, we obtain $8/27 = (2/3)^m$ whence by inspection $m = 3$. Then $k = \frac{1}{4}$, and the required function is $4y = x^3$.

*Solution* 2. Substitution of the logarithms of the given coördinates into (17) gives

$$\begin{cases} 1.20412 = K + 0.60206m \\ 1.73239 = K + 0.77815m. \end{cases}$$

Solving, $m = 3$ and $K = -.60206 = 9.39794 - 10$, $k = .25$.

**15. Parabolic Trend.** Data of broad economic or social significance extending over a long period of years may often be described by an arc of a second degree parabola. The equation of a parabola is of the form

$$y = \alpha + \beta x + \gamma x^2$$

where $\alpha$, $\beta$, $\gamma$ are the parameters to be determined.

If the $x$'s are equispaced we may let

$$t = \frac{x - \bar{x}}{c},$$

where $\bar{x} = (x_1 + x_N)/2$ and $c = |x_{i+1} - x_i|$, and thereby effect considerable simplification in evaluating the constants. In $t$ and $y$ coördinates the equation will, of course, involve different constants and we may write its equation in the form

$$y = A + Bt + Ct^2. \tag{18}$$

The method of moments may again be used and since (18) is a polynomial this method also gives the best fitting curve in a least squares sense. Because there are three constants to be determined we must equate the second moments as well as the zero*th* and first moments. Imposing these conditions of moments between the observed and computed ordinates, we obtain the three normal equations:

$$\begin{aligned} \sum y &= NA + B\sum t + C\sum t^2 \\ \sum ty &= A\sum t + B\sum t^2 + C\sum t^3 \\ \sum t^2y &= A\sum t^2 + B\sum t^3 + C\sum t^4. \end{aligned}$$

Since the mean is chosen as origin $\sum t = 0$. With this choice of origin and because the $x$'s are equispaced it can be shown that $\sum t^3 = 0$. Therefore the normal equations simplify into

$$\text{(19)} \qquad \begin{cases} B = \dfrac{\sum ty}{\sum t^2} \\ AN + C\sum t^2 = \sum y \\ A\sum t^2 + C\sum t^4 = \sum t^2 y. \end{cases}$$

When the summations involved in these equations are evaluated from the data the values of $A$, $B$, and $C$ can easily be determined.

*Example* 11. Fit a parabola to the following data.

NUMBER OF DIVORCES PER 1000 MARRIAGES IN THE UNITED STATES 1900–1930

| Year | $y$ | $x$ | $t$ | $ty$ | $t^2$ | $t^2y$ | $t^4$ |
|---|---|---|---|---|---|---|---|
| 1900 | 81 | 0 | −3 | −243 | 9 | 729 | 81 |
| 1905 | 84 | 5 | −2 | −168 | 4 | 336 | 16 |
| 1910 | 88 | 10 | −1 | − 88 | 1 | 88 | 1 |
| 1915 | 104 | 15 | 0 | 0 | 0 | 0 | 0 |
| 1920 | 134 | 20 | 1 | 134 | 1 | 134 | 1 |
| 1925 | 148 | 25 | 2 | 296 | 4 | 592 | 16 |
| 1930 | 170 | 30 | 3 | 510 | 9 | 1530 | 81 |
| Sums | 809 | $\bar{x} = 15$ | | 441 | 28 | 3409 | 196 |

From (19),

$$B = \frac{441}{28}$$

$$7A + 28C = 809$$

$$28A + 196C = 3409.$$

Solving the last two equations simultaneously we obtain,

$$A = \frac{322}{3}, \qquad C = \frac{173}{84}.$$

Therefore,

$$y = \frac{322}{3} + \frac{441}{28}t + \frac{173}{84}t^2.$$

If we desire the equation in the original form we substitute $t = \frac{1}{5}(x - 15)$ and obtain

$$y = \frac{322}{3} + \frac{441}{28}\left(\frac{x - 15}{5}\right) + \frac{173}{84}\left(\frac{x - 15}{5}\right)^2$$

which simplifies into

$$y = 78.62 + .68x + .0824x^2.$$

Upon the hypothesis that divorces will continue to increase according to this trend, we may estimate the number for 1950 for example. When $x = 50$ in the above equation, we find $y = 318.62$.

**16. The Gompertz Curve.** The curve which bears his name was suggested in 1825 by Gompertz for use in actuarial science. Recently it has had some application as a *growth curve* in business and population forecasting and in certain problems in education. Its equation[1] is

$$y = kg^{c^x}. \tag{20}$$

To determine the parameters, we first transform (20) into the logarithmic form

$$Y = K + Gc^x \tag{20a}$$

where $Y = \log y$, $K = \log k$, $G = \log g$. The number, $N$, of observations available must be such that $N = 3n$ where $n$ is the number in each of three subgroups with no observations omitted; that is, $N$ must be divided into three blocks of data consisting of $n$ items each. It is also necessary that the values of the independent variable $x$ be equispaced. Then the origin can be chosen so that $x$ takes the values $0, 1, 2, \cdots, 3n - 1$. If these values of $x$ are substituted in (20a) we obtain the three sets of functional $Y$'s shown in (a), (b), and (c).

$$\left.\begin{array}{ll} 0 & Y_0 \\ 1 & Y_1 \\ \cdots & \cdots \\ n-1 & Y_{n-1} \end{array}\right\} \quad \sum_{i=0}^{n-1} Y_i \quad \left\{\begin{array}{l} Y_0 = K + Gc^0 \\ Y_1 = K + Gc \\ \cdots\cdots \\ Y_{n-1} = K + Gc^{n-1} \end{array}\right\} \quad (a)$$

$$\left.\begin{array}{ll} n & Y_n \\ n+1 & Y_{n+1} \\ \cdots & \cdots \\ 2n-1 & Y_{2n-1} \end{array}\right\} \quad \sum_{i=n}^{2n-1} Y_i \quad \left\{\begin{array}{l} Y_n = K + Gc^n \\ Y_{n+1} = K + Gc^{n+1} \\ \cdots\cdots \\ Y_{2n-1} = K + Gc^{2n-1} \end{array}\right\} \quad (b)$$

$$\left.\begin{array}{ll} 2n & Y_{2n} \\ 2n+1 & Y_{2n+1} \\ \cdots & \cdots \\ 3n-1 & Y_{3n-1} \end{array}\right\} \quad \sum_{i=2n}^{3n-1} Y_i \quad \left\{\begin{array}{l} Y_{2n} = K + Gc^{2n} \\ Y_{2n+1} = K + Gc^{2n+1} \\ \cdots\cdots \\ Y_{3n-1} = K + Gc^{3n-1} \end{array}\right\} \quad (c)$$

[1] For a derivation see *Mathematical Theory of Life Insurance* — Forsyth. John Wiley and Sons, Inc.

Let $S_1$, $S_2$, $S_3$ denote respectively the totals of the subgroups $(a)$, $(b)$, and $(c)$. Thus we have

$$S_1 = nK + G(1 + c + \cdots + c^{n-1})$$
$$S_2 = nK + Gc^n(1 + c + \cdots + c^{n-1})$$
$$S_3 = nK + Gc^{2n}(1 + c + \cdots + c^{n-1}).$$

Then

$$S_2 - S_1 = G(c^n - 1)(1 + c + \cdots + c^{n-1})$$
$$S_3 - S_2 = Gc^n(c^n - 1)(1 + c + \cdots + c^{n-1})$$

whence we obtain

$$c^n = \frac{S_3 - S_2}{S_2 - S_1}.$$

Writing the expression for $S_2 - S_1$ in the form

$$S_2 - S_1 = G\frac{(c^n - 1)^2}{c - 1}$$

and solving for $G$, we obtain

$$G = \frac{(S_2 - S_1)(c - 1)}{(c^n - 1)^2}.$$

The expression for $S_1$ may be written

$$S_1 = nK + \frac{G(1 - c^n)}{1 - c};$$

so we have

$$K = \frac{1}{n}\left[S_1 - \frac{G(1 - c^n)}{1 - c}\right].$$

In the above expressions, $S_1$, $S_2$, $S_3$ denote sums of the functional $Y$'s. If these are now replaced by the empirical data so that

$$S_1 = \sum_{n}^{2n-1} Y_i, \quad S_2 = \sum_{2n}^{3n-1} Y_i, \quad S_3 = \sum_{3n}^{4n-1} Y_i,$$

where $Y_i$ refers to the observed $Y$'s, then $c$ can be determined from the expression for $c^n$. Using the value of $c$, $G$ can be determined, and then $K$.

If $c < 1$, it is clear from $(20a)$ that $Y \to K$ as $x \to \infty$. Then $y = k$ is an asymptote and $k$ is sometimes called the *ceiling* of the curve. (See Figure 32.)

For an application of the above method to a problem in business, see *Statistical Methods (Revised Edition)* by Mills, page 672.

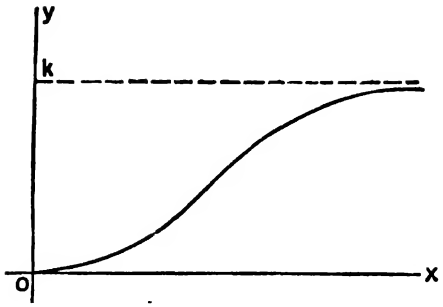



FIG. 32

**17. Remarks and References.** The methods of least squares and moments do not select the appropriate curve. They merely determine the "best" values of the parameters in the equation of the curve which has been selected previously to describe the observed data. The question of the type of curve which should be fitted to the data is not always easy to answer. The selection of the appropriate mathematical function depends to a large extent upon the investigator's experience in the field in which the problem lies and his knowledge of the properties of curves. It always helps to plot the data first. The usual requirements for practical purposes are that (*a*) the curve must represent well the trend of the empirical data, and (*b*) the mathematical expression must not involve too many parameters and those present must be calculable from the data. In dealing with time series, if the objective is to find out what would happen if the *percentage* change should continue as it has on the average in the past, then an exponential trend is indicated. If the objective is to find out what would happen if the yearly (or monthly, etc.) change should continue as it has in the past, a straight line trend is indicated.

We will merely mention here two other important curves which require more advanced mathematics in their treatment. The *logistic*, or so-called Reed-Pearl curve, is used extensively in studying various growth phenomena. Its function is of the form

$$y = \frac{1}{a + bc^x}$$

and it resembles somewhat the Gompertz curve discussed above. For further discussion of this curve and methods of fitting it see

1. *Elements of Statistics* — Davis and Nelson.
2. *Statistical Methods, Revised* — Mills.

The function $y = ks^x g^{c^x}$

is known as Makeham's law. It is used in actuarial work. The student having a working knowledge of the calculus will find an interesting discussion of its use in the field of insurance in an article entitled *Makeham's Laws of Mortality*, Rietz, American Mathematical Monthly, vol. 28, p. 471.

The logistic curve was used in studies on the rate of growth of the population of the United States. But its usefulness in this connection fell somewhat short, apparently,[1] of the claims of its sponsors. Two other references relating to the population of our country may appropriately be mentioned here. Although they do not involve problems of curve fitting they do afford instructive examples of the application of scientific method to social and political problems. They are

1. *Bibliography on Methods of Apportionment in Congress* — E. V. Huntington. *American Mathematical Monthly*, vol. 49 (1942), pp. 115-117.
2. *Determination of the Center of Population in the United States. School Science and Mathematics*, May and June, 1942.

## Exercises

1. If the rate of change of $y$ with respect to $x$ is always proportional to the attained value of $y$ then $y$ is what kind of a function of $x$?
2. Determine $A$ and $B$ in the best fitting curve of the type (13) for the following data.

| Data | | Form for Computations | | | |
|---|---|---|---|---|---|
| $x$ | $y$ | $t$ | $Y$ | $tY$ | $t^2$ |
| 0 | 1000 | | | | |
| 5 | 100 | | | | |
| 10 | 10 | | | | |
| 15 | 1 | | | | |
| 20 | .1 | | | | |

3. (*a*) Prove formula (11).
   (*b*) Graph the curve $y = 10e^{-2x}$.
4. Find the best fitting parabola for the following points: $(-4, 2)$, $(0, 8)$, $(4, 9)$, $(8, 11)$, $(12, 8)$, $(16, 5)$. *Ans.* $y = 7.2 + .94x - .07x^2$.

[1] *Differential Equations Subject to Error, and Population Estimates* — Harold Hotelling. *Jour. Amer. Stat. Assoc.*, vol. 22 (1927), pp. 283-314.

5. If the values of $t$ form an arithmetic progression and $\sum t$ 0 prove that $\sum t^3 = 0$.
6. ($a$) Add the values $x = 30$, $y = 37$ to the data of Example 6 and find the trend line. *Ans.* $y = .8x + 10.43$.
   ($b$) On the hypothesis that the apparent trend continues, predict the value of $y$ when $x = 35$
7. In a tensile test of a metal bar the following observations were made, where $x$ represents the load in tons and $y$ the elongation in ten-thousandths of an inch:

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 14 | 27 | 40 | 55 | 68 |

Determine a linear relation between $x$ and $y$ by the theory of least squares.

8. In the following table $y$ represents the fire losses in the United States in millions of dollars. Taking the origin of $x$ at 1915 find the best fitting line, in a least squares sense, for the data.

| $x$ | 1915 | 1917 | 1919 | 1921 | 1923 | 1925 |
|---|---|---|---|---|---|---|
| $y$ | 172 | 290 | 321 | 495 | 535 | 570 |

9. ($a$) Add the values $x = 6$, $y = 300$ to the data of Example 7 (p. 153) and find the equation of the best fitting exponential curve.

$$\textit{Ans.} \quad Y = .4617x - .2534$$
$$y = .56e^{1.06x}.$$

   ($b$) Plot the given data and the curve obtained in ($a$) on semi-log paper.
10. Distinguish between the forms of the curves represented by the functions $y = Ae^{-Bx}$ and $y = Ke^{-hx^2}$ where $A$, $B$, $K$, and $h$ are positive real numbers. If these functions were plotted on semi-log paper what kind of curves would be obtained?
11. Determine by inspection the value of ($a$) $10^{\log_{10} e}$, ($b$) $a^{\log_a N}$.
12. Solve for $x$: $\log_{10}(x^2) = (\log_{10} x)(\log_e x)$.
13. Solve for $x$: $\log_{10}(x^2) - \log_{10}(x/10) = 2$.
14. Determine a number $x$ such that the square of $\log x$ exceeds $\log x$ by 2. (Logs to base 10. Two answers.)
15. On semi-logarithmic coördinate paper, a straight line is drawn through the points (2, 1) and (4, 100). Determine the function which has that line as its graph. *Hint.* Use the form $y = Ar^x$. *Ans.* $100y = 10^x$.
16. Same as exercise 15 for the points (1, 6) and (2, 18). *Ans.* $y = 2(3^x)$.

**17.** On logarithmic coördinate paper, a straight line is drawn through the points (2, 12) and (3, 27). Determine the function which has that line as its graph. *Ans.* $y = 3x^2$.

**18.** Data from a certain experiment involving voltage ($v$) as a function of time ($t$) are plotted on logarithmic coordinate paper, and are found to exhibit a linear trend there. A line is drawn, with a transparent ruler, which seems to fit the plotted data best. Two points on this line are (6, 18) and (8, 32). Determine an equation expressing $v$ in terms of $t$ whose logarithmic graph is the line.

**19.** Draw the graph of $y = 25x^n$ on logarithmic coördinate paper, ($a$) when $n = 2$, ($b$) when $n = -2$. Mark scales clearly.

**20.** The graph of $y = \log_{10} x$ assists one in remembering several important properties of the logarithms of real numbers. Sketch this graph and state some of these properties.

**21.** Read and report on one or more of the references cited in §17.

*Note.* Source material for additional exercises on curve fitting may be found in the current volumes of the following publications:

1. *Statistical Abstract of the United States.*
2. *World Almanac and Book of Facts.*

# CHAPTER VIII

## CORRELATION THEORY

**1. The Meaning of Simple Correlation.** So far we have been concerned with the problems which arise from variation in a single variable. We will now consider the simultaneous variation of two variables. Methods for disclosing the facts of co-variation and for measuring the degree of relationship existing between two variables are due mainly to the English biometricians Sir Francis Galton (1822–1911) and Karl Pearson (1857–1936).

Data presenting two sets of related measurements or observations may arise in many fields of activity yielding $N$ pairs of corresponding variates $(x_i, y_i)$, $i = 1, 2, 3, \cdots, N$. Thus $x$ may represent July rainfall and $y$ the average yield of corn in a certain section; $x$ may be an index of commodity prices and $y$ an index of employment over the same period; we may be interested in a group of school children in which $x$ is their height and $y$ their weight, or $x$ may refer to their reading ability and $y$ to their spelling ability; we may be studying the chance distributions which are obtained in throwing two dice where $x$ is the number obtained in throws of a single die and $y$ is the number obtained in throws of the two dice together.

*Example* 1. In the following set of selected heights (inches), $x$ = stature of father, $y$ = stature of son.

| $x$ | 69 | 70 | 69 | 68 | 70 | 73 | 69 | 67 | 69 | 64 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 68 | 69 | 72 | 67 | 70 | 71 | 72 | 66 | 71 | 65 |

*Example* 2. (*Snedecor.*) The following data on twelve trees are adapted from the results of an experiment to test the phenomenon that the injury by codling moth larvae seems to be greatest on apple trees bearing a small crop. Here $x$ = hundreds of fruit on a tree, $y$ = percentage of fruits wormy.

| $x$ | 15 | 15 | 12 | 26 | 18 | 12 | 8 | 38 | 26 | 19 | 29 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 52 | 46 | 38 | 37 | 37 | 37 | 34 | 25 | 22 | 22 | 20 | 14 |

When the given pairs of values are represented by dots locating the points whose rectangular coördinates are $(x, y)$ we obtain a so-called " scatter diagram " (Figure 33). The problem is to determine the degree of association, or correlation as it is called, between the $x$'s and the corresponding $y$'s since this indicates the significance of the relationship.

The field of correlation may be thought of as bounded on the one extreme by perfect functional dependence and on the other extreme by complete independence in the probability sense. For example, the pairs of values which satisfy the equation $y = 2x - 5$ do not present a statistical problem. In this case the relationship is defined by a mathematical function $y = f(x)$. Similarly, at the other extreme we would not be concerned with pairs of values which are completely independent in the probability sense, as, for example, the grades of students in statistics and the heights of their fathers. Two variables are said to be statistically related when they lie between these two extremes of relationship.

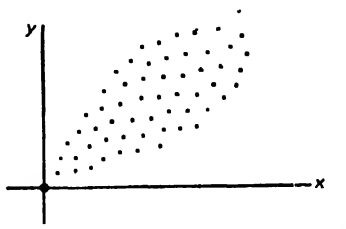


FIG. 33

The theory of correlation is concerned with a twofold problem: first with measuring the indicated relationship, and secondly with predicting or estimating the average value of $y$ associated with a designated value of $x$.

**2. The Coefficient of Correlation.** It is fairly obvious from Figure 33 that with values of $x$ in an assigned interval $\Delta x$ ($\Delta x$ small) the corresponding values of $y$ differ considerably. There is said to be positive correlation if, for an assigned $x$ larger than $\bar{x}$, the mean of the corresponding $y$ values is larger than $\bar{y}$, and, for values of $x$ smaller than $\bar{x}$, the mean of the corresponding values of $y$ is less than $\bar{y}$. On the other hand, as $x$ increases the tendency may be for $y$ to decrease. In this case, for an assigned $x$ larger than $\bar{x}$ the mean of the corresponding $y$ values is less than $\bar{y}$, and for an assigned $x$ less than $\bar{x}$ the mean of the corresponding $y$'s is greater than $\bar{y}$. There is then said to be negative correlation. If, for an assigned $x$ taken at random a corresponding $y$ is no more likely to be above than below $\bar{y}$, the variables are independent in the statistical or probability sense and there is said to be zero correlation between them.

When the variables are correlated there is a tendency for the dots

**Theorem I.** *The value of r is independent of the origin of reference and the units of measurement.*

*Proof:* Let

$$u = \frac{x - x_0}{h}, \qquad v = \frac{y - y_0}{k}.$$

Then

$$x = uh + x_0, \qquad y = vk + y_0, \qquad \sigma_x = h\sigma_u, \qquad \sigma_y = k\sigma_v.$$

Substituting in (2) we obtain

$$r = \frac{\frac{1}{N}\sum(u - \bar{u})(v - \bar{v})}{\sigma_u \sigma_v} \tag{4}$$

$$= \frac{\frac{1}{N}\sum uv - \bar{u}\bar{v}}{\sigma_u \sigma_v}, \tag{4a}$$

where $\sigma_u = \left[\frac{1}{N}\sum u^2 - \bar{u}^2\right]^{1/2}$, $\sigma_v = \left[\frac{1}{N}\sum v^2 - \bar{v}^2\right]^{1/2}$.

Since (4) and (4*a*) are independent of the constants $x_0$, $y_0$, $h$, and $k$, the theorem is proved.

This property is of fundamental importance. It means that the units of measurement for the two sets of observed quantities can be chosen independently of each other. If the two sets of quantities are of the same kind, the units need not be the same in both cases; and, what is more important, if the quantities are of different kinds, so that the units are not comparable at all, the coefficient $r$ nevertheless may have a definite meaning. (Of course the value of the coefficient will be affected by a change in the method of measurement of one of the quantities, such as the substitution of an area for a length in estimating the size of an object, or the assignment of different relative weights to the questions on an examination.)

The pairs $(x_i, y_i)$ may be all distinct or there may be repetitions among them. But it is necessary to impose the condition that neither $x_i$ nor $y_i$ shall be constant throughout. This condition is imposed to insure that the denominator shall not vanish in the various formulas for $r$. *The Algebra of Correlation* — Dunham Jackson. *American Mathematical Monthly*, vol. 31 (1924), pp. 110–121.

When the given values of $x$ and $y$ are large and a computing machine is not available, the computations may be lightened by an appropriate choice of these constants. If only the origin of reference is changed,

then $h = k = 1$, and $u = x - x_0, v = y - y_0$. If the means are taken as the origin of reference by letting $x' = x - \bar{x}$ and $y' = y - \bar{y}$, then $\bar{x}' = \bar{y}' = 0$ and the formula becomes,

$$r = \frac{\frac{1}{N}\sum x'y'}{\left[\frac{1}{N}\sum x'^2\right]^{1/2}\left[\frac{1}{N}\sum y'^2\right]^{1/2}}. \tag{5}$$

A subscript notation should be attached to $r$ when there are several series of variates. Thus, $r_{xy}$ for the $(x, y)$ series, $r_{xz}$ for the $(x, z)$ series, $r_{12}$ for the series denoted by $(x_1, x_2)$, etc.

*Example* 3. To illustrate the formulas we will compute the value of $r$ for the following data. Here $x$ = Brokers' Loans in billions of dollars and $y$ = *The Annalist's* index of the prices of fifty rail and industrial stocks in 1929. We choose $u = x - 5.00$ and $v = y - 250$.

| *Month* | $x$ | $y$ | $u$ | $v$ | $uv$ | $u^2$ | $v^2$ |
|---|---|---|---|---|---|---|---|
| *J* | 5.33 | 248 | .33 | −2 | −0.66 | .1089 | 4 |
| *F* | 5.67 | 248 | .67 | −2 | −1.34 | .4489 | 4 |
| *M* | 5.65 | 243 | .65 | −7 | −4.55 | .4225 | 49 |
| *A* | 5.56 | 249 | .56 | −1 | −.56 | .3136 | 1 |
| *M* | 5.53 | 235 | .53 | −15 | −7.95 | .2809 | 225 |
| *J* | 5.28 | 265 | .28 | 15 | 4.20 | .0784 | 225 |
| *J* | 5.77 | 282 | .77 | 32 | 24.64 | .5929 | 1024 |
| *A* | 6.02 | 303 | 1.02 | 53 | 54.06 | 1.0404 | 2809 |
| *S* | 6.35 | 290 | 1.35 | 40 | 54.00 | 1.8225 | 1600 |
| *O* | 6.80 | 230 | 1.80 | −20 | −36.00 | 3.2400 | 400 |
| *N* | 4.88 | 201 | −.12 | −49 | 5.88 | .0144 | 2401 |
| *D* | 3.45 | 206 | −1.55 | −44 | 68.20 | 2.4025 | 1936 |
| Sums | | | 6.29 | 0 | 159.92 | 10.7659 | 10678 |
| $\frac{1}{N}$ Sums | | | .5242 | 0 | 13.3267 | .8972 | 889.8333 |

*Computations:* $\sigma_u = [.8972 - (.5242)^2]^{1/2} = .79$

$\sigma_v = [889.8333]^{1/2} = 29.83.$

From (4*a*) we have,

$$r = \frac{13.3267}{(29.83)(.79)} = .57.$$

Experienced computers use calculating machines to great advantage in large-scale computational studies. The following reference is recommended to students who expect to engage in such work: "The Calculation of Correlation Coefficients from Ungrouped Data"—P. S. Dwyer, *Journal of the American Statistical Association*, vol. 35 (1940), pp. 671–673.

## Exercises

1. When $x'$ and $y'$ represent deviations from the means,
   (*a*) Show from (1) that $\sum x'y' = Nr\sigma_x\sigma_y$.
   (*b*) Show that $N\sigma_x^2 = \sum x'^2$.
2. Derive formula (3) from (2).
3. Show that (3) may be written as

$$r = \frac{N\sum xy - \sum x \sum y}{[\{N\sum x^2 - (\sum x)^2\}\ \{N\sum y^2 - (\sum y)^2\}]^{1/2}}.$$

4. Find $r$ for the data of Example 1.
5. Find $r$ for the data of Example 2.
6. The following data represent the ages of husband ($x$) and wife ($y$) of twenty couples. Find $r$ using (5). *Ans.* 0.856.

| $x$ | 22 | 24 | 26 | 26 | 27 | 27 | 28 | 28 | 29 | 30 | 30 | 30 | 31 | 32 | 33 | 34 | 35 | 35 | 36 | 37 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 18 | 20 | 20 | 24 | 22 | 24 | 27 | 24 | 21 | 25 | 29 | 32 | 27 | 27 | 30 | 27 | 30 | 31 | 30 | 32 |

7. In studying a set of pairs of related variates, a statistician has completed the preliminary arithmetic and obtained the following results:
   $N = 100$; $\sum x^2 = 1{,}585{,}000$; $\sum x = 12{,}500$; $\sum xy = 1{,}007{,}425$; $\sum y^2 = 648{,}100$; $\sum y = 8{,}000$. Find $\bar{x}$, $\bar{y}$, $\sigma_x$, $\sigma_y$, $r$.
8. The table in Exercise 2, page 97, contains the grades made on two tests by twenty-five students in mathematics. Find $r$ for these data. *Ans.* 0.786.
9. Suggest examples of negative correlation.
10. In the following anthropometric measurements on a random sample of twenty male freshmen, taken from the Physical Education Department,

| $x$ | $y$ | $z$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|
| 68.5 | 33.6 | 148 | 65.3 | 33.0 | 136 |
| 67.2 | 35.0 | 144 | 65.1 | 34.0 | 144 |
| 67.7 | 30.2 | 145 | 64.8 | 37.3 | 170 |
| 63.8 | 30.0 | 108 | 69.6 | 33.4 | 154 |
| 69.9 | 33.0 | 130 | 68.2 | 31.5 | 122 |
| 64.7 | 31.0 | 112 | 68.8 | 32.0 | 141 |
| 68.4 | 33.0 | 134 | 72.3 | 35.0 | 159 |
| 66.4 | 30.2 | 112 | 67.8 | 33.7 | 134 |
| 69.1 | 33.3 | 143 | 71.3 | 31.5 | 136 |
| 71.0 | 32.3 | 136 | 63.5 | 33.6 | 126 |

$x$ represents height, $y$ represents chest measurement, both measurements being taken to the nearest tenth of an inch, and $z$ represents weight to the nearest pound. Find the coefficient of correlation ($a$) between $x$ and $y$, ($b$) between $x$ and $z$, ($c$) between $y$ and $z$.

**4. Regression.** The properties of $r$ can be studied by fitting a line to the scatter diagram in such a way as to make the sum of the squares of the vertical distances from the points to the line a minimum.

When such a line is referred to the point $(\bar{x}, \bar{y})$ as origin, we have seen (§9, Chapter VII) that its equation is $y' = m_1 x'$ where

$$m_1 = \frac{\sum x'y'}{\sum x'^2}$$

and $x' = x - \bar{x}$, $y' = y - \bar{y}$. This value of $m_1$ may easily be expressed in terms of $r$ and the standard deviations, as follows:

$$m_1 = \frac{Nr\sigma_y\sigma_x}{N\sigma_x^2} = r\frac{\sigma_y}{\sigma_x}.$$

Therefore, the equation of our line, referred to a system of axes whose origin is at the means of the variates, is

$$y' = \frac{\sigma_y}{\sigma_x} rx'. \tag{6}$$

This is called the *regression line of $y$ on $x$.* The term *regression* was used first by Galton in studying inheritance of stature. He found that offspring of abnormally tall or short parents tend to "step back" or "regress" to the ordinary population height. However, as now used, regression line has no reference to biometry, but is merely a convenient term.

By fitting a line $x' = m_2 y'$ to the points of the scatter diagram in such a way that the sum of the squares of the horizontal distances from the points to the line shall be a minimum, it is possible to deduce a second regression line (the regression line of $x$ on $y$) whose equation, referred to $(\bar{x}, \bar{y})$, is

$$x' = \frac{\sigma_x}{\sigma_y} ry'. \tag{7}$$

Note that (7) cannot be obtained by solving for $x'$ in (6). The two regression lines will coincide if, and only if, $r = \pm 1$. From

the equations of the regression lines it is evident that if $r > 0$, an increase in the one variable tends to accompany an increase in the other; if $r < 0$, an increase in the one will be accompanied by a decrease in the other.

Equations (6) and (7) are usually expressed in terms of the original variables $x$ and $y$ instead of the deviations $x'$ and $y'$. It is obvious that they may be written as

$$(8) \qquad y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

and

$$(9) \qquad x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

when referred to the origin of $x$ and $y$.

Equation (8) may be used to estimate values of $y$ corresponding to designated values of $x$. Similarly, from equation (9) we may estimate $x$ for designated values of $y$. It would be appropriate to use (8) as a predicting equation when the variation in $y$ is caused or controlled by the variation in $x$; (9) would be used when the variation in $x$ is caused or controlled by the variation in $y$.

The quantity $m_1 = r(\sigma_y/\sigma_x)$ is called the regression coefficient of $y$ on $x$, being the variation in $y$ corresponding to a unit change in $x$. Likewise, $m_2 = r(\sigma_x/\sigma_y)$ is called the regression coefficient of $x$ on $y$. Thus the numerical value of $r$ is given by $(m_1 m_2)^{1/2}$ but its sign must be that which is common to the two regression coefficients. The following quotation from Snedecor (reference 13, list p. 6) sheds light on the distinction between regression and correlation.

> The point of interest here is that $r$ is the geometric mean of the two regression coefficients. In ordinary units of measurement, therefore, $r$ is an average of the two regression coefficients used in (i) estimating $y$ from $x$ and (ii) estimating $x$ from $y$. This serves to clarify the relation of the two coefficients, correlation and regression, in measuring relationship. The latter is the appropriate one if one variable, $y$, may be designated as dependent on the other, $x$. Values of $y$ may be partly controlled or caused by $x$, as when the available amounts of some glandular secretion cause differences in the sizes of organisms. Or, $y$ may be subsequent to $x$, as weight gain in nutrition experiments follows the measurement of initial weight. In such cases, the regression of $y$ on $x$ is usually the statistic that furnishes the information desired. It is then appropriate to attempt to estimate the value of $y$ from a knowledge of the corresponding value of $x$. Correlation, on the other hand, is the appropriate measure of the relation between

two variates like statures of husband and wife. The two heights are known to be associated through some complex of social and biological causes, but neither may be looked upon as a consequence of the other. In this sense correlation is a two-way average of relationship, while regression is directional. Of course, there are many variables whose relationship may be studied by means of either correlation or regression, or both. It is necessary only to keep clearly in mind the character of the relation being considered.

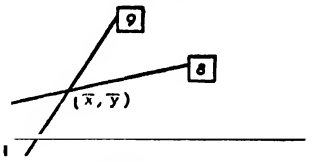



FIG. 35

Geometrically, $m_1$ is the slope of line (8) and $1/m_2$ is the slope of line (9). The two lines intersect at $(\bar{x}, \bar{y})$.

## Exercises

1. Derive the equation of the line of regression of $x$ on $y$ as suggested above.
2. Find the equations of both lines of regression for Exercise 6 (page 176), and plot them. *Ans.* $y = .888x - .64$
   $x = .825y + 8.55.$
3. Using the appropriate equation, find the estimated values of $y$ corresponding to the given values of $x$, for Exercise 6 (page 176).
4. Given the following results for the heights and weights of 1000 men students:
   $\bar{y} = 68.00$ in., $\bar{x} = 150.00$ lbs., $r = .60$,
   $\sigma_y = 2.50$ in., $\sigma_x = 20.00$ lbs.
   John Doe weighs 200 lbs., Richard Roe is five feet tall.
   Estimate the height of Doe from his weight, and the weight of Roe from his height.
   *Ans.* Doe's height = 71.75 in.
   Roe's weight = 111.6 lbs.
5. (*a*) Given the following:
   $\sum x = 150{,}000$, $\sum x^2 = 22{,}725{,}000$, $\sum xy = 10{,}522{,}500$,
   $\sum y = 70{,}000$, $\sum y^2 = 4{,}936{,}000$, $N = 1000$.
   Find $\bar{x}$, $\bar{y}$, $\sigma_x$, $\sigma_y$, $r$, and the lines of regression.
   (*b*) Suppose the data in (*a*) refer to the weight in pounds ($x$) and the height in inches ($y$) of a sample of 1000 policemen. Suppose Paul Private weighs 160 pounds and Saul Sergeant is 6 feet tall. Estimate the height of Private and the weight of Sergeant.

**5. The Standard Error of Estimate.** The average concentration of the points around the regression line of $y$ on $x$ may be measured by the expression $\frac{1}{N}\sum d^2$ where $d$ is the difference between an observed $y$ and the $y$ obtained from the regression line. The value of

$\frac{1}{N}\sum d^2$ will be denoted by $S_y^2$, and $S_y$ is called the standard deviation of the errors of estimate, or more briefly the *standard error of estimate.* The errors of estimate are the deviations of the observed values of $y$ from the corresponding estimated $y$'s. Or to describe them another way, they are the deviations of the sample $y$'s from the assumed population $y$'s. It can be shown that $S_y^2 = \sigma_y^2(1 - r^2)$. To prove this we may write the sum of the squares of the deviations in the form:

$$NS_y^2 = \sum\left(y' - r\frac{\sigma_y}{\sigma_x}x'\right)^2 = \sum y'^2 - 2r\frac{\sigma_y}{\sigma_x}\sum x'y' + r^2\frac{\sigma_y^2}{\sigma_x^2}\sum x'^2$$
$$= N\sigma_y^2 - 2Nr^2\sigma_y^2 + Nr^2\sigma_y^2 = N\sigma_y^2(1 - r^2).$$

Hence, we have

(10) $$S_y^2 = \sigma_y^2(1 - r^2)$$

and

(10a) $$S_y = \sigma_y(1 - r^2)^{1/2}.$$

An analogous consideration of the differences between the $x$'s and the regression line of $x$ on $y$ gives for the square of the standard error of estimate of the $x$'s

(11) $$S_x^2 = \sigma_x^2(1 - r^2).$$

**6. Properties of the Correlation Coefficient and Standard Error of Estimate.** Certain properties of $r$ may now be deduced. It is obvious from (10) that $|r| \leq 1$ because both the left member and $\sigma_y^2$ are positive or zero. Therefore,

$$-1 \leq r \leq 1.$$

If the points all lie exactly on the regression line, the left member of (10) vanishes and $r = \pm 1$. There is then said to be perfect linear correlation, since the relation between $x$ and $y$ is given exactly by a linear function. A large numerical value of $r$ means that the regression lines are close to coincidence and the points in a scatter diagram cluster closely around the regression lines.

When the regression lines (8) and (9) are expressed in standard units, they become respectively

(12) $$t_y = rt_x$$

and

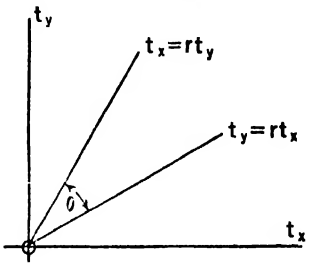
(13) $$t_x = rt_y \quad \text{or} \quad t_y = \frac{1}{r} t_x$$

where

$$t_x = \frac{x - \bar{x}}{\sigma_x} \quad \text{and} \quad t_y = \frac{y - \bar{y}}{\sigma_y}.$$

In this form we see at once that as one variable $t_x$ increases, the other variable $t_y$ increases (or decreases) to an extent that depends upon $r$. Thus $r$ measures co-variation in the variables when they are expressed in comparable units and when regression is linear.

In standard units, $r$ is the slope of line (12) and $1/r$ is the slope of line (13). When $r = 0$, the regression equations become $t_y = 0$ and $t_x = 0$ in standard units or $y = \bar{y}$ and $x = \bar{x}$ in the original units. These are also the equations of the coördinate axes. Therefore, when $r = 0$ the regression lines are perpendicular to each other and coincide with the $t_x$ and $t_y$ axes. When $r = 1$ the regression equations become identical and the two lines coincide in quadrants I and III. Similarly, when $r = -1$ they coincide in quadrants II and IV. In each case the coincident lines bisect the quadrants if the equations are expressed in standard units, but not otherwise unless $\sigma_y = \sigma_x$. The angle $\theta$ between the regression lines varies from 0° to 90° as $r$ varies from one to zero.




When there is no correlation between $x$ and $y$ then $r = 0$, and the variables are said to be independent in the statistical sense. On the other hand, when $r = 0$, it is not necessarily true that the variables are statistically independent. Indeed there may be a high correlation[1] with non-linear regression when $r = 0$. (Non-linear regression will be considered in §21.) Incidentally, the phrase "independent variables" in the statistical sense should not be confused with the phrase "independent variables" which is used in the ordinary sense of analysis to designate the variables on which a specified function depends. However, the two usages, though quite distinct, are not fundamentally contradictory, since functional dependence can be regarded as a limiting case of statistical dependence.

[1] See H. L. Rietz, *On Functional Relations for which the Coefficient of Correlation is Zero. Journal American Statistical Association*, vol. 16, 1919, pp. 472–476.

For an appreciation of the use of $S_y$ in passing judgment upon the precision to be expected in estimating values of $y$ by means of the regression equation of $y$ on $x$, it is instructive to consider simultaneously the meanings of (8) and (10$a$) as $|r|$ varies from 0 to 1. When $r = 0$, (8) becomes $y = \bar{y}$ which means that the best estimate of $y$ for *any* value of $x$ is the mean of the $y$-distribution. In other words, knowledge of $x$ is of no value in predicting $y$. When $r = 0$ in (10$a$), $S_y = \sigma_y$. This is to be expected since the dispersion $S_y$ about the line $y = \bar{y}$ is the same as the dispersion $\sigma_y$ of the given $y$'s about their mean. But as $|r|$ increases from 0 to 1, $S_y$ decreases from $\sigma_y$ to 0. Graphically, the meaning of this improvement in $S_y$ in comparison

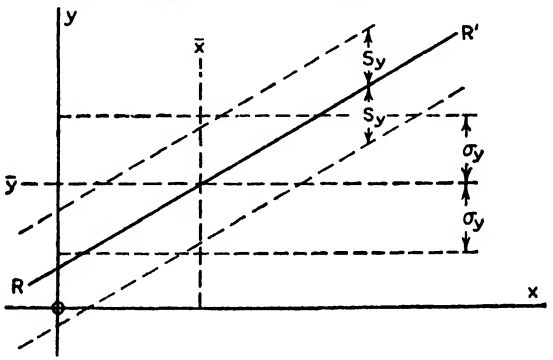
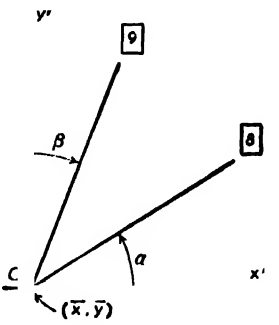



FIG. 36 — FOR A FIXED VALUE OF $\sigma_y$, $S_y$ DECREASES IN PROPORTION TO $(1 - r^2)^{1/2}$ AS $r$ INCREASES

with $\sigma_y$, as $r$ increases, is shown in Figure 36 where parallel lines are drawn at a vertical distance of $S_y$ on either side of the regression line $RR'$. For a given value of $|r| \neq 0$ this strip encloses the average dispersion about the line. The strip on either side of $y = \bar{y}$ at a distance of $\sigma_y$ from it encloses the average dispersion about the line when $r = 0$. As $|r|$ increases from 0, the line rotates from the horizontal position of $y = \bar{y}$ to the terminal position it would have when $|r| = 1$, and at the same time $S_y$ decreases toward 0. Formula (10$a$) tells us that as $|r|$ thus increases, $S_y$ decreases from $\sigma_y$ in proportion to $(1 - r^2)^{1/2}$.

A similar analysis could be made concerning the line of regression (9) of $x$ on $y$ which rotates from the vertical position $x = \bar{x}$ when $|r| = 0$ to meet and coincide with line (8) when $|r| = 1$. As line (9) rotates, $S_x$ decreases from $\sigma_x$ to 0 in proportion to $(1 - r^2)^{1/2}$ as $r$ increases.

As $|r| \to 1$, (12) and (13) rotate toward each other at equal angular velocities. When they are coincident their slope is $\pm 1$. Lines (8) and (9) rotate at angular velocities which are proportional to $m_1 = \tan \alpha$ and $m_2 = \tan \beta$, respectively, when $m_1$ and $m_2$ are defined in §4. Their slope at coincidence is $\pm \sigma_y/\sigma_x$. For line (12) it can be shown that




$$\frac{1}{N}\sum \delta^2 = 1 - r^2 \tag{14}$$

where $\delta$ is the difference between an observed value of $t_y$ and the ordinate obtained from (12) for the corresponding value of $t_x$. Thus,

$$\begin{aligned}\frac{1}{N}\sum \delta^2 &= \frac{1}{N}\sum (t_y - rt_x)^2 \\ &= \frac{1}{N}\sum t_y^2 - \frac{2r}{N}\sum t_x t_y + \frac{r^2}{N}\sum t_x^2 \\ &= 1 - 2r^2 + r^2 \\ &= 1 - r^2.\end{aligned}$$

This result would also be apparent from the derivation of (10) since $\delta = d/\sigma_y$ where $d$ refers to residuals in units other than standard units.

It is obvious from (14) that the maximum value of $\frac{1}{N}\sum \delta^2$ is unity. Therefore, adopting

$$1 - \frac{1}{N}\sum \delta^2 \tag{15}$$

as a measure of goodness of fit, we see from (14) and (15) that $r^2$ is a measure of the goodness of fit of (12) to the points of the scatter diagram expressed in standard units. By an analogous argument a similar conclusion concerning (13) can be made.

**7. Further Discussion.** Given a set of $N$ pairs of $x$ and $y$ correlated values. Suppose the necessary constants are evaluated to obtain the regression equation (8). Then if the given values of $x$ are substituted in this equation, a set of estimated $y$'s, say $_E y$, will be

obtained. The mean, ${}_E\bar{y}$, of these estimated $y$'s is the same as the mean of the observed $y$'s. The proof is as follows. From (8) we have

$$ {}_Ey = \bar{y} + r\frac{\sigma_y}{\sigma_x}(x - \bar{x}). $$

Then

$$ {}_E\bar{y} = \frac{1}{N}\sum_1^N \bar{y} + r\frac{\sigma_y}{\sigma_x}\frac{1}{N}\sum_1^N (x_i - \bar{x}). $$

But $\sum_1^N (x_i - \bar{x}) = 0$ by Theorem VI, Chapter III. So ${}_E\bar{y} = \bar{y}$.

We now state the following theorem.

**Theorem II.** *The variance, $\sigma_{E_y}{}^2$, of the estimated $y$'s equals $r^2\sigma_y{}^2$.*

*Proof:* By definition,

$$ \sigma_{E_y}{}^2 = \frac{1}{N}\sum ({}_Ey_i - {}_E\bar{y})^2. $$

From the above discussion, $({}_Ey_i - {}_E\bar{y})$ is the same as $(y_i - \bar{y})$ which is given by (8). So

$$ \sigma_{E_y}{}^2 = \frac{1}{N}\sum_1^N \left[ r\frac{\sigma_y}{\sigma_x}(x_i - \bar{x}) \right]^2 $$

$$ = r^2\frac{\sigma_y{}^2}{\sigma_x{}^2}\frac{1}{N}\sum^N (x_i - \bar{x})^2. $$

Hence

$$ \sigma_{E_y}{}^2 = r^2\sigma_y{}^2. \tag{16} $$

From this theorem and (10) we obtain

$$ S_y{}^2 = \sigma_y{}^2 - \sigma_{E_y}{}^2. \tag{17} $$

This relation helps to clarify the meaning of $r$ and of $S_y$. It is conventional to call $\sigma_{E_y}{}^2$ the variance in $y$ which can be *explained* from knowledge of $x$; that is, which the regression of $y$ on $x$ accounts for. (In the language of some writers, $\sigma_{E_y}{}^2$ measures the variation of regression about the mean.) Therefore, (17) shows that $S_y{}^2$ is the variation in $y$ after the accompanying variation in $x$ is duly discounted. $S_y{}^2$ is sometimes called the *residual* variance because it measures the variation in the dependent variable $y$ which knowledge of $x$ fails to account for. This relation can be depicted geometrically by the sides of a right triangle. To standardize the representation we can take $\sigma_y = 1$ as the diameter of a semicircle within which

is inscribed the right triangle, as in Figure 37. In the figure, $\cos\theta = \sigma_{Ey}/\sigma_y$. So from (16) we have $\cos\theta = r$. The particular values of $\theta$ in the figure, found from a table of cosines, are $\theta = 36° 52'$ when $r = .8$, and $\theta = 25° 50'$ when $r = .9$. When $r = 1$, then $\sigma_{Ey} = \sigma_y$ and the regression of $y$ on $x$ accounts for all the variation in $y$.

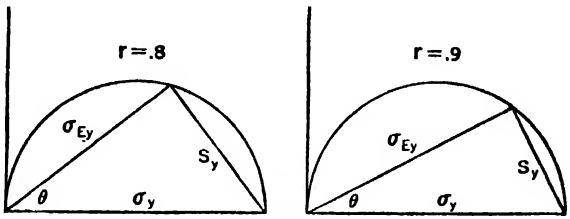



FIG. 37

**Theorem III.** *The correlation between observed and estimated values of $y$ is the same as that between the observed values of $x$ and $y$.*

*Proof:* We are to show that

$$\frac{\frac{1}{N}\sum y_E y - \bar{y}_E \bar{y}}{\sigma_{Ey}\sigma_y}$$

reduces to one of the formulas for $r$. Substituting the values for $_Ey$, $_E\bar{y}$, $\sigma_{Ey}$ into the above expression and simplifying, we obtain (3). The details of the proof are left to the student as an exercise.

**8. Coefficient of Alienation.** A measure of the failure to improve estimates of $y$ from knowledge of correlation is given by

$$k = (1 \quad r^2)^{1/2}. \tag{18}$$

It is sometimes called the coefficient of alienation. Incidentally, it is interesting to observe that the functional relation between $k$ and $r$ is shown, graphically, by a semicircle of unit radius, *i.e.*,

$$f(r) = (1 - r^2)^{1/2}.$$

The formula

$$k' = 1 - (1 - r^2)^{1/2}$$

may be called the *improvement factor* because it shows the decrease in $S_y/\sigma_y$ as $|r|$ increases. It is clear that

$$k^2 = \frac{1}{N}\sum\delta^2 = \frac{S_y^2}{\sigma_y^2} \quad \text{and } k' = 1 - k.$$

Table 31 gives[1] values of $k$ and $k'$ for values of $r$. With no knowledge of correlation, the best estimate of an individual $y$ is $\bar{y}$. Values of $k'$ for assigned $r$'s show how much better than this guess is the estimate of an individual $y$ value with knowledge of correlation. For example, when $r = .5$ the column headed $k$ in Table 31 shows that the standard error $S_y$ is about 87% of $\sigma_y$. Or, from the $k'$ column, $S_y$ has been reduced only 13% from what it would have been if $\bar{y}$ had been used for prediction purposes. The third column thus shows how the prediction value of $r$ varies with $r$. Thus as $|r|$ decreases from 1 to .8, $S_y/\sigma_y$ increases from 0 to 60%. Or from another point of view, as $|r|$ increases from 0 to .8, the error of estimate is improved by only 40%. A correlation of $r = .9$ permits prediction of individual $y$'s only 56% better than a mere guess based on the mean.

It is fairly obvious that we cannot, with any considerable degree of reliability, predict from ordinary values of $r$ an individual $y$ for an assigned $x$. However, with a large $N$, we can give a very reliable prediction of the mean of $y$ values that correspond to an assigned value of $x$. This can best be explained from a correlation table which is used when $N$ is large and which will be explained in the next section.

TABLE 31 — VALUES OF $r$ AND THE CORRESPONDING VALUES OF $k$ AND $k'$

| $r$ | $k$ | $k'$ |
|---|---|---|
| .1 | .995 | .005 |
| .2 | .980 | .020 |
| .3 | .954 | .046 |
| .4 | .917 | .083 |
| .5 | .866 | .134 |
| .6 | .800 | .200 |
| .7 | .714 | .286 |
| .8 | .600 | .400 |
| .9 | .436 | .564 |
| .92 | .392 | .608 |
| .94 | .341 | .659 |
| .96 | .280 | .720 |
| .98 | .198 | .811 |
| 1.00 | 0.000 | 1.000 |

[1] Constructed from a table of sines and cosines. Letting $r = \cos\theta$, $\sin\theta = (1 - \cos^2\theta)^{1/2} = (1 - r^2)^{1/2}$.

## Exercises

1. Given the following correlated data:

| $x$ | 8 | 6 | 4 | 7 | 5 |
|---|---|---|---|---|---|
| $y$ | 9 | 8 | 5 | 6 | 2 |

(*a*) Compute the correlation coefficient.
(*b*) Find the regression line of $y$ on $x$.
(*c*) Find the estimated values of $y$ corresponding to the given values of $x$.
(*d*) Compute the standard error $S_y$ of predictions in two different ways.
*Ans.*

$$r = \frac{2.4}{\sqrt{2}\sqrt{6}} = .69, \qquad m_1 = r\frac{\sqrt{6}}{\sqrt{2}} = 1.2, \qquad S_y = \sqrt{3.12} = 1.76.$$

*Note.* In practical work, it is never worth while calculating a correlation coefficient for so few observations. These fictitious data are given solely as an exercise on which the student can test his knowledge of the methodology.

2. Prove that the ratio of variance of the estimated $y$'s (taken about their mean) to the variance $\sigma_y^2$ of the given $y$'s is equal to $r^2$.
3. If $S_y^2/\sigma_y^2 = 1 - r^2$ is the percentage of the total variance of $y$ uncontrolled by knowledge of $x$, what is the remaining percentage, determined by or calculable from knowledge of $x$?
4. What equation is the equivalent mathematical statement for the following words?

   If the respective deviations in each series, $x$ and $y$, from their means were expressed in units of standard deviations — that is, if each were divided by the standard deviation of the series to which it belongs — and plotted to a scale of standard deviations, the slope of a straight line best describing the plotted points would be the correlation coefficient $r$.
5. Given the standard deviations $\sigma_x$ and $\sigma_y$ of two distributions of correlated variates:
   (*a*) What is the standard error in estimating $y$ from $x$ if $r = 0$?
   (*b*) By how much is $S_y$ in (*a*) reduced if $r$ is increased to .25?
   (*c*) How large must $r$ be in order that $S_y$ be one-half as large as in (*a*)?
   (*d*) What must $r$ be in order that $S_y$ be reduced to one-third its value in (*a*)?
   (*e*) At what value of $r$ is $S_y$ reduced to zero?
   (*f*) For any value of $r$, what is the ratio between the standard error of estimating $y$ from $x$ and the standard deviation of the $y$-distribution?
6. Evaluate the following statements:
   (*a*) A correlation coefficient less than zero indicates an absence of linear relationship.
   (*b*) A correlation coefficient of $r = .6$ indicates twice as close relationship as a coefficient of $r = .3$.

7. If all the points lie exactly on the regression line of $y$ on $x$, show that $S_y^2 = 0$ and hence that $r = \pm 1$.

8. Show that $S_y^2$ may be computed by means of the relation

$$NS_y^2 = \sum y'^2 - \frac{(\sum x'y')^2}{\sum x'^2}$$

where the primes denote deviations from the means.

9. (*For analytics students.*) Show that the tangent of the angle from line (8) to line (9) is

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\left\{\frac{1 - r^2}{r}\right\}$$

and from line (12) to line (13) is

$$\tan\theta' = \frac{1 - r^2}{2r}$$

What is the value of $\theta$ when $r = 1$; when $r = 0$?

10. The least-squares criterion of best fit requires that $\sum\delta^2$ be a minimum, where $\delta$ is the distance between the line and a point. Three cases arise depending on whether

*Case I*, $\delta$ is measured parallel to the $y$-axis,
*Case II*, $\delta$ is measured parallel to the $x$-axis,
*Case III*, $\delta$ is measured perpendicular to the line.

We have seen that Case I yields line (12) and that Case II yields line (13). In Case III the line has no universally accepted name but it may be called the "geometrically best-fitting line."

(*For calculus students.*) For Case III prove the following:

($a$) In standard units, the equation of the line is

$$t_y = t_x \quad \text{if} \quad r > 0$$

and

$$t_y = -t_x \quad \text{if} \quad r < 0.$$

*Solution.* Let the equation of the required line be

$$t_y = mt_x + k.$$

Then by analytics,

$$\frac{1}{N}\sum\delta^2 = \frac{1}{N}\sum\left(\frac{mt_x + k - t_y}{\sqrt{1 + m^2}}\right)^2$$

$$= \frac{m^2 + k^2 + 1 - 2mr}{1 + m^2}.$$

To make this a minimum, first put $k^2 = 0$. Call the result $f(m)$. Then

$$f(m) = \frac{m^2 + 1 - 2mr}{1 + m^2},$$

$$f'(m) = \frac{2m^2r - 2r}{(1 + m^2)^2},$$

$$f''(m) = \frac{4mr(3 - m^2)}{(1 + m^2)^3}.$$

The second derivative will be positive when $m$ and $r$ have the same sign. Since $f(m)$ is a minimum when $m = \pm 1$, we are to take $m = 1$ when $r > 0$ and $m = -1$ when $r < 0$.

(*b*) If $r = 0$, all lines (for which $k^2 = 0$) fit equally well. *Hint.* If $r = 0$, $f(m) = 1$.

(*c*) $\frac{1}{N}\sum\delta^2 = 1 - |r|$. *Hint.* What is the value of $f(m)$ when $m = \pm 1$?

Note that $|r| = +r$, if $r > 0$ and $|r| = -r$, if $r < 0$.

(*d*) Goodness of fit is measured by $|r|$.

(*e*) When $r = .6$ the fit is twice as good as when $r = .3$.

**11.** The following query and answer appeared in *Biometrics Bulletin*, vol. 1, no. 3, pp. 36–37. "Research" assignment: Investigate the references cited in the answer and justify the procedure which is recommended (under the given hypothesis).

*Query.* A problem that has bothered me is the fitting of regression lines when their position is restricted in some way. For example, suppose a test is made of the relationship between the number of fish caught in a body of water and the average number which can be caught out of it, with a standard amount of fishing. In fitting a regression line to such data, we know that the point (0, 0) must fall on the line, since if no fish are present certainly none will be caught. In other words, we have one point which is free from sampling error. The unique importance of this point will, it seems to me, make observations in its neighborhood of relatively less importance than observations at a distance from it, where there is no fixed guide-post. Do you know of any treatment of situations of this sort, by which the best straight (or curved) line could be fitted to data where there is one point which *must* be satisfied? The standard deviation from regression ("standard error of estimate") and the standard error of the regression would also be available. Or are these concepts pertinent in such a question?

*Answer.* Deming (§15 and §11 of reference 4) gives both a general method and some particular solutions of your problem. Snedecor (reference 6) opens his Chapter 6 with an illustration of the simple case in which $x$ is measured without error and the variance of $y$ is constant for all values of $x$.

Observations in the neighborhood of (0, 0) may or may not be of less importance than those at greater distances; it depends on the variance of $y$. One often finds that this variance increases with $x$. In fact, there are many situations in which it seems reasonable to suppose that in the sampled population the standard deviation of $y$ is directly proportional to $x$. If you think this hypothesis is suitable in your fishing, the appropriate method is to calculate the ratios $x/y$ where $x$ is the number of fish caught and $y$ is the total number of fish, then apply to them the statistical procedure suitable for a single variate. — George W. Snedecor.

**9. Correlation Table.** When the sample to be studied is large, it is more convenient to replace the scatter diagram by a correlation

table. We may divide the $xy$-plane into rectangles of convenient size, and all points of the scatter diagram falling within any rectangle are thought of as being concentrated at the center of this rectangle. A number is then written within the rectangle to designate the number of points at its center. A correlation table is therefore a two-way frequency table exhibiting the frequencies in each class interval.

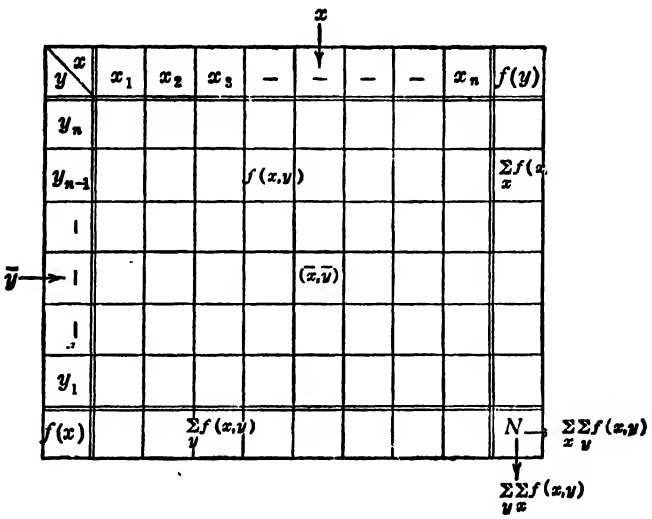
TABLE 32

| | | 65 – 69 | 70– 74 | 75– 79 | 80– 84 | 85– 89 | 90– 94 | 95– 99 | |
|---|---|---|---|---|---|---|---|---|---|
| | $y$ \ $x$ | 67 | 72 | 77 | 82 | 87 | 92 | 97 | $f(y)$ |
| 90–94 | 92 | | | | 1 | 2 | 3 | 1 | 7 |
| 85–89 | 87 | | | 1 | 3 | 8 | 1 | 5 | 18 |
| 80–84 | 82 | 4 | 4 | 6 | 4 | 9 | 1 | | 28 |
| 75–79 | 77 | 3 | 3 | 7 | 6 | 4 | | | 23 |
| 70–74 | 72 | 2 | 3 | 5 | 6 | 1 | 1 | | 18 |
| 65–69 | 67 | 3 | 2 | | | | | | 5 |
| 60–64 | 62 | 1 | | | | | | | 1 |
| | $f(x)$ | 13 | 12 | 19 | 20 | 24 | 6 | 6 | 100 |

Suppose Table 32 is constructed in this way for a set of average daily grades ($x$) and final examination grades ($y$) of 100 students. When the data have been thus grouped into classes, the class marks are regarded as the variate values. Thus in Table 32 there are 9 students whose daily grades are 87 and whose final examination grades are 82. The last column labeled $f(y)$ represents the distribution of $y$ variates and the last row labeled $f(x)$ represents the distribution of $x$ variates. A correlation table is thus a bivariate distribution. In Table 32 the width of the class interval is the same for $x$ and $y$, but of course this is not generally the case.

**10. Notation.** In order to compute $r$ from a correlation table it will be necessary to develop new notation. Since we are now dealing with frequencies in both the $x$-direction and the $y$-direction, we will distinguish between them by $f(x)$ and $f(y)$. To be sure, this has the disadvantage of being the same symbol as that for function, but from the context no ambiguity should arise.

Generalizing, a correlation table is of the following form:




The rectangles containing the frequencies are called cells. The frequency in a typical cell is denoted by $f(x, y)$, meaning the frequency in the cell whose coördinates are $x$ and $y$, where $x$ and $y$ are the mid-values of the class intervals. Both columns and rows are sub-distributions of the total frequency $N$. Each column is a frequency distribution of $y$'s corresponding to a mid-$x$ value. Similarly, each row is a frequency distribution corresponding to a mid-$y$ value. The sum along any row is denoted by $\sum_x f(x, y)$, being the sum of the frequencies in the $(x, y)$ cells in the $x$-direction. Since the marginal total for any row is the total frequency corresponding to a given value of $y$, it is therefore written in the column headed $f(y)$. Thus, in Table 32, for $y = 92$,

$$\sum_x f(x, y) = \sum_x f(x, 92) = 1 + 2 + 3 + 1 = 7.$$

Similarly, $\sum_y f(x, y)$ denotes a summation in the $y$-direction of all the entries in a column, corresponding to a fixed value of $x$, so it denotes an entry in the bottom row which contains the $f(x)$ frequencies. Thus, for $x = 67$

$$\sum_y f(67, y) = 4 + 3 + 2 + 3 + 1 = 13.$$

Summarizing,

$$(19) \qquad \sum_x f(x, y) = f(y); \qquad \sum_y f(x, y) = f(x).$$

With regard to $N$, we may obtain it from a correlation table in three ways: (1) by adding the entries across the rows and then totaling the resulting sums in the marginal column labeled $f(y)$; (2) by adding the entries along the columns and then totaling the results in the marginal row labeled $f(x)$; (3) by adding the entries in the cells in any order whatsoever. Hence, the following notation,

$$(20) \qquad \sum_y \sum_x f(x, y) = \sum_x \sum_y f(x, y) = \sum_{x, y} f(x, y) = N,$$

will denote, respectively, the above-named procedures or orders in summing. From (19) and (20) we have

$$(21) \qquad N = \sum_y f(y) = \sum_x f(x) = \sum_{x, y} f(x, y).$$

We may call $f(x)$ and $f(y)$ the marginal distributions of $x$ and $y$, respectively. A correlation table with cell frequencies $f(x, y)$ uniquely determines the marginal totals $f(x)$ and $f(y)$. The converse, however, is false. For example, we might replace the four cell frequencies in the upper right-hand corner of Table 32 by the cell frequencies $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 4 \\ \hline \end{array}$ without disturbing the marginal totals.

**11. Means and Variances.** We will now express the means in terms of this notation, taking first the mean of $x$'s. From the fundamental definition, we must multiply each $x$ by its corresponding frequency in the cells and sum the results, taking the products in any order whatsoever. Hence,

$$\bar{x} = \frac{1}{N} \sum_{x, y} x f(x, y).$$

This may also be written

$$\bar{x} = \frac{1}{N} \sum_x \sum_y x f(x, y) = \frac{1}{N} \sum_x x \sum_y f(x, y) = \frac{1}{N} \sum_x x f(x).$$

Observe that the $x$ may be moved to the left of $\sum_y$ in the second expression because $x$ is treated as a constant in a summation performed with respect to $y$.

Similarly, we have,

$$\bar{y} = \frac{1}{N}\sum_{x,y} yf(x, y) = \frac{1}{N}\sum_{y}\sum_{x} yf(x, y)$$

$$= \frac{1}{N}\sum_{y} y\sum_{x} f(x, y) = \frac{1}{N}\sum_{y} yf(y).$$

The student will observe that the last expression for the mean in each case is identical with that given for a frequency distribution of one variable, when allowance is made for the necessity of distinguishing between variables.

Any column is an $x$ array of $y$'s, so the symbol $\bar{y}_x$ is appropriate for the mean of a column. Similarly, $\bar{x}_y$ denotes the mean of a $y$ array of $x$'s, *i.e.*, of a row. We may now state the following theorem.[1]

**Theorem IV.** *The mean $\bar{y}$ for the whole table (in the y-direction) is equal to the mean of the values $\bar{y}_x$ for the several columns when each $\bar{y}_x$ is weighted with the frequency in that column.*

*Proof:* We are required to show that

$$\frac{1}{N}\sum_{x} f(x)\bar{y}_x = \bar{y}$$

where

$$\bar{y}_x = \frac{1}{f(x)}\sum_{y} yf(x, y).$$

Upon substituting in the first equation the value of $\bar{y}_x$ as given by the second equation, we have

$$\frac{1}{N}\sum_{x}\sum_{y} yf(x, y) = \frac{1}{N}\sum_{x,y} yf(x, y) = \bar{y}.$$

It is suggested that the student state and prove a similar theorem concerning $\bar{x}$.

In this new notation, the definitions of the variances becomes

$$\sigma_x^2 = \frac{1}{N}\sum_{x,y}(x - \bar{x})^2 f(x, y)$$

$$= \frac{1}{N}\sum_{x} x^2 f(x) - \bar{x}^2;$$

[1] This is actually the same as Theorem IX on page 45, but it seems worthwhile to state and prove it in the new notation.

$$\sigma_y^2 = \frac{1}{N}\sum_{x,\,y}(y - \bar{y})^2 f(x, y)$$

$$= \frac{1}{N}\sum_{y} y^2 f(y) - \bar{y}^2.$$

### Exercises

1. Evaluate the following expressions in Table 32.

   (*a*) For $x = 82$,

   $$\sum_y f(x, y), \qquad \sum_y y f(x, y), \qquad f(x), \qquad \bar{y}_x.$$

   (*b*) For $y = 87$,

   $$\sum_x f(x, y), \qquad \sum_x x f(x, y), \qquad f(y), \qquad \bar{x}_y.$$

2. Refer to Table 27 (Chapter V) and let $x$ be the number of a column. Express the answers in the third and second lines from the bottom of the table in terms of the notation of this section. Thus for $x = 1$,

   $$\bar{y}_x = \frac{1}{f(x)}\sum_y y f(x, y) = \frac{1}{7}[85 + (75)2 + (65)2 + (55)2] = 67.86.$$

**12. Computation of Means.** Just as in the case of a one-way frequency distribution it was found convenient to choose an arbitrary origin and take the class interval as the unit, so we now do likewise. Let

$$u = \frac{1}{h}(x - x_0); \qquad i.e., \qquad x = uh + x_0. \tag{22}$$

Hence,

$$\bar{x} = \bar{u}h + x_0 \tag{23}$$

where

$$\bar{u} = \frac{1}{N}\sum_u u f(u).$$

Likewise, let

$$v = \frac{1}{k}(y - y_0); \qquad i.e., \qquad y = vk + y_0 \tag{24}$$

whence

$$\bar{y} = \bar{v}k + y_0, \tag{25}$$

where

$$\bar{v} = \frac{1}{N}\sum_v v f(v).$$

Then a suitable form for computing the means of the $x$'s and $y$'s is as follows:

| | $u$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 | $f(y)$ = $f(v)$ | $vf(v)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | $y$ \ $x$ | 67 | 72 | 77 | 82 | 87 | 92 | 97 | | |
| 3 | 92 | | | | 1 | 2 | 3 | 1 | 7 | 21 |
| 2 | 87 | | | 1 | 3 | 8 | 1 | 5 | 18 | 36 |
| 1 | 82 | 4 | 4 | 6 | 4 | 9 | 1 | | 28 | 28 |
| 0 | 77 | 3 | 3 | 7 | 6 | 4 | | | 23 | 0 |
| −1 | 72 | 2 | 3 | 5 | 6 | 1 | 1 | | 18 | −18 |
| −2 | 67 | 3 | 2 | | | | | | 5 | −10 |
| −3 | 62 | 1 | | | | | | | 1 | −3 |
| $f(x)=f(u)$ | | 13 | 12 | 19 | 20 | 24 | 6 | 6 | 100 | 54 |
| $uf(u)$ | | −39 | −24 | −19 | 0 | 24 | 12 | 18 | −28 | |

*Computations:*

$$\bar{u} = \frac{1}{N}\sum uf(u) = \frac{-28}{100} = -.28,$$

whence

$$\bar{x} = 82 + 5(-.28) = 80.6.$$

$$\bar{v} = \frac{1}{N}\sum vf(v) = .54,$$

whence

$$\bar{y} = 77 + 5(.54) = 79.7.$$

In the table $f(v) = f(y)$ and $f(u) = f(x)$ because $u$ and $v$ are merely different ways of describing the cells but in no way change the frequencies in those cells.

**13. Computation of $r$.** In the expressions of §10 and §11 the $(u, v)$ coördinates could have been used instead of $(x, y)$. The use of the former simplifies the computation of $r$. A preliminary discussion of certain expressions will help in understanding the formula for $r$ to be used for a correlation table. Let us consider first the following expression:

$$(a) \qquad \sum uvf(u, v).$$

This means: multiply the $f$ in each cell by the $u$ and $v$ coördinates of that cell and add the results, proceeding from cell to cell over the whole table in any order whatsoever. But it may be more con-

venient to proceed in a definite order, say down the columns. Then (*a*) becomes

$$(b) \qquad \sum_u \sum_v uvf(u, v) = \sum_u u \sum_v vf(u, v).$$

The expression $\sum_v vf(u, v)$ in the right member of (*b*) means: for any $u$ (*i.e.*, for any column), multiply each $f$ by its own $v$ and add the results. Let us denote this sum by $V$. Then the right member of (*b*) means: multiply the $V$ for each column by the $u$ of that column and add the results, proceeding from column to column (*i.e.*, summing in the $u$-direction). We may also obtain the same result as in (*a*) by proceeding along the rows. Thus (*a*) may be written

$$(c) \qquad \sum_v \sum_u uvf(u, v) = \sum_v v \sum_u uf(u, v).$$

The expression $\sum_u uf(u, v)$ means: for any $v$ (*i.e.*, for any row), multiply each $f$ in the row by its own $u$ and add the results. If we call this sum $U$, then the right member of (*c*) means: multiply the $U$ for each row by the $v$ for that row and add the results, proceeding from row to row (*i.e.*, summing in the $v$-direction).

We are now ready to derive the formula for $r$.

Since we are now dealing with a frequency distribution, the fundamental definition of $r$ becomes

$$(26) \qquad r = \frac{\frac{1}{N}\sum_{x,y}(x - \bar{x})(y - \bar{y})f(x, y)}{\sigma_x \sigma_y}.$$

From (22) and (23), we have

$$(x - \bar{x}) = h(u - \bar{u}),$$

and from (24) and (25),

$$(y - \bar{y}) = k(v - \bar{v}).$$

Since $(x, y)$ and $(u, v)$ are merely different notations for the same cell, we have

$$f(x, y) = f(u, v).$$

For computing purposes, the standard deviations are defined as

follows:

$$\sigma_x = h\sigma_u \qquad \sigma_y = k\sigma_v \tag{27}$$

where

$$\begin{cases} \sigma_u = \left[\dfrac{1}{N}\sum_u u^2 f(u) - \bar{u}^2\right]^{1/2} \\ \sigma_v = \left[\dfrac{1}{N}\sum_v v^2 f(v) - \bar{v}^2\right]^{1/2}. \end{cases} \tag{28}$$

Therefore, (26) becomes

$$r = \frac{\dfrac{1}{N}\sum_{u,v}(u-\bar{u})(v-\bar{v})f(u,v)}{\sigma_u\sigma_v}.$$

If now we let

$$U = \sum_u uf(u,v) \qquad \text{and} \qquad V = \sum_v vf(u,v),$$

then since

$$\sum_{u,v} uvf(u,v) = \sum_v v\sum_u uf(u,v) = \sum_u u\sum_v vf(u,v),$$

the above expression for $r$ may be written in either of the following ways:

$$r = \frac{\dfrac{1}{N}\sum_v vU - \bar{u}\bar{v}}{\sigma_u\sigma_v} \tag{29}$$

$$= \frac{\dfrac{1}{N}\sum_u uV - \bar{u}\bar{v}}{\sigma_u\sigma_v}.$$

The fact that

$$\sum_v vU = \sum_u uV$$

serves as a check in the table.

The above procedure is illustrated in Table 35.

*Explanation:* The table is self-explanatory except possibly the $U$ and $V$ entries. Recalling that $U = \sum_u uf(u,v)$, the first entry in the $U$ column is obtained from the sum of the following products: $0{\cdot}1 + 1{\cdot}2 + 2{\cdot}3 + 3{\cdot}1 = 11$; the second entry from $-1{\cdot}1 + 0{\cdot}3 + 1{\cdot}8 + 2{\cdot}1 + 3{\cdot}5 = 24$. Since $V = \sum vf(u,v)$ the first

TABLE 35 — COMPUTATION OF $r$ FOR DATA OF TABLE 32

| | $u$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 | $f(v)$ | $vf(v)$ | $v^2f(v)$ | $U$ | $vU$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | $x$ / $y$ | 67 | 72 | 77 | 82 | 87 | 92 | 97 | | | | | |
| 3 | 92 | | | | 1 | 2 | 3 | 1 | 7 | 21 | 63 | 11 | 33 |
| 2 | 87 | | | 1 | 3 | 8 | 1 | 5 | 18 | 36 | 72 | 24 | 48 |
| 1 | 82 | 4 | 4 | 6 | 4 | 9 | 1 | | 28 | 28 | 28 | −15 | −15 |
| 0 | 77 | 3 | 3 | 7 | 6 | 4 | | | 23 | 0 | 0 | −18 | 0 |
| −1 | 72 | 2 | 3 | 5 | 6 | 1 | 1 | | 18 | −18 | 18 | −14 | 14 |
| −2 | 67 | 3 | 2 | | | | | | 5 | −10 | 20 | −13 | 26 |
| −3 | 62 | 1 | | | | | | | 1 | −3 | 9 | −3 | 9 |
| $f(u)$ | | 13 | 12 | 19 | 20 | 24 | 6 | 6 | 100 | 54 | 210 | | (115) |
| $uf(u)$ | | −39 | −24 | −19 | 0 | 24 | 12 | 18 | −28 | | | | |
| $u^2f(u)$ | | 117 | 48 | 19 | 0 | 24 | 24 | 54 | 286 | | | | |
| $V$ | | −7 | −3 | 3 | 7 | 30 | 11 | 13 | | | | | |
| $uV$ | | 21 | 6 | −3 | 0 | 30 | 22 | 39 | (115) | | | | |

Check

entry in the $V$ row is obtained from $1\cdot4 + 0\cdot3 + -1\cdot2 + -2\cdot3 + -3\cdot1 = -7$. Similarly, for the other entries.

*Computations:*

$$\sigma_u^2 = \frac{1}{N}\sum u^2 f(u) - \bar{u}^2 = 2.86 - (-.28)^2$$

$$= 2.7816.$$

$$\sigma_u = \sqrt{2.7816} = 1.67.$$

$$\sigma_v^2 = \frac{1}{N}\sum v^2 f(v) - \bar{v}^2 = 2.10 - (.54)^2$$

$$= 1.8084.$$

$$\sigma_v = \sqrt{1.8084} = 1.34.$$

Therefore from (29) we have

$$r = \frac{1.15 - (-.28)(.54)}{(1.67)(1.34)} = 0.58.$$

**14. Remarks on Computation of $r$.** (*a*) *Sign of $r$.* It should be observed that the sign of $r$ depends on the choice of the positive direction along each coördinate axis. In Table 35 the origin of reference is chosen so that the data occur in the first quadrant and the directions on the $(x, y)$-axes are the conventional ones. These directions were preserved in changing to $(u, v)$ coördinates. If we had reversed the direction of the $v$-axis by labeling the $y$ values larger than $y = 77$ by $v = -1, -2, -3$, and those less than $y = 77$ by $v = 1, 2, 3$, the sign of $r$ would be changed. But if the directions of *both* $u$ and $v$ were reversed, the sign of $r$ would be unchanged.

(*b*) *Grouping errors.* When $N$ is small, say less than 100, and the data are grouped into cells, grouping errors are introduced. In general, the fewer cells used, the greater the errors. These may be corrected, in part, by applying Sheppard's corrections to $\sigma_u$ and $\sigma_v$. However, this will not be insisted upon in this course.

(*c*) *Commercial charts.* Computations can be expedited by the use of commercially prepared correlation charts. Several types of chart are available on the market. In her book (reference 15), Professor Helen M. Walker explains the merits of two of these which are recommended. She also gives the following advice to beginners: "A chart is not a crutch to help the novice. It is a means of speeding up operations after they are well understood."

### Exercises

1. By equation (29), show that $r$ is independent of the choice of origin and of the units of measurement.
2. In Table 35, evaluate the following sums:

$$\sum_u f(u, 2), \quad \sum_v f(2, v), \quad \sum_u uf(u, 1), \quad \sum_v vf(-2, v), \quad \sum_u \sum_v vf(u, v), \quad \sum_{u,v} uvf(u, v)$$

$$\frac{1}{f(v)} \sum_u f(u, v) \quad \text{if } v = 0.$$

3. Derive (29).
4. For the table on page 200, find $r$ and $\bar{x}$, $\bar{y}$, $\sigma_x$, $\sigma_y$. Note that $x_0$, $y_0$, $h$ and $k$, do not need to be determined to compute $r$, but are required for the means and standard deviation of $x$ and $y$.

**15. Regression Lines for a Correlation Table.** The data of a correlation table may be thought of as dots lying many deep at the centers of the several cells. There are, of course, $f(x, y)$ of these in any cell whose coördinates are $(x, y)$, and $f(x)$ is the total number of dots in a vertical column whose coördinate is $x$. Suppose now we

HEIGHTS AND WEIGHTS OF 200 FRESHMEN

(Heights to Nearest $\frac{1}{10}$ Inch; Weights to Nearest $\frac{1}{2}$ Pound)

| $y$ \ $x$ | 90–99.5 | 100– | 110– | 120– | 130– | 140– | 150– | 160– | 170– | 180– | 190– | 200–209.5 | $f(y)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 76–77.9 | | | | 1 | | | | | | | | | 1 |
| 74– | | | | | | | 1 | 1 | 1 | 1 | | | 4 |
| 72– | | | | 1 | 1 | 1 | 4 | | 1 | | | | 8 |
| 70– | | | 1 | 2 | 6 | 7 | 6 | 2 | 1 | 2 | 1 | 1 | 29 |
| 68– | | | 2 | 8 | 17 | 8 | 9 | 2 | 1 | 1 | 1 | | 49 |
| 66– | | | 8 | 16 | 14 | 13 | 6 | 2 | 1 | | | 1 | 61 |
| 64– | | 3 | 8 | 7 | 7 | 3 | 3 | 1 | 1 | | | | 33 |
| 62– | 1 | 4 | 1 | 7 | 1 | | | | | | | | 14 |
| 60– | | | | | | | | | | | | | 0 |
| 58–59.9 | | 1 | | | | | | | | | | | 1 |
| $f(x)$ | 1 | 8 | 20 | 42 | 46 | 32 | 29 | 8 | 6 | 4 | 2 | 2 | 200 |

*Ans.* $\bar{x} = 138.45$ lbs.; $\bar{y} = 67.82$ in.
$\sigma_x = 19.6$ lbs.; $\sigma_y = 2.8$ in.
$r = 0.48$.

replace all the data in each column by an equal number of data concentrated at the mean of that column. If we denote the ordinate of this mean point by $\bar{y}_x$, we have

$$\bar{y}_x = \frac{1}{f(x)} \sum_y y f(x, y). \tag{30}$$

Hence, $\bar{y}_x f(x)$ represents the totality of all the values in a column.

For each of the columns there will be a value of (30). Taking the hypothesis that the mean points of the several columns lie approximately on a straight line $\bar{y}_x = m_1 x + k$, we may find $m_1$ and $k$ under a least-squares criterion of approximation. If, in applying the criterion,

the square of the difference between the observed mean, ${}_c\bar{y}_x$, and the computed mean, ${}_c\bar{y}_x$, for each array, *viz.*, $(\bar{y}_x - m_1x - k)^2$, is weighted with the number $f(x)$ in the array, it turns out that we get the same values for $m_1$ and $k$ which we obtained when we fitted the regression line of $y$ on $x$ to the scatter diagram.

In proving this, the student of calculus[1] would have an easy task in obtaining the normal equations:

$$(31) \qquad \begin{cases} \sum_x (\bar{y}_x - m_1x - k)f(x) = 0 \\ \sum_x (\bar{y}_x - m_1x - k)xf(x) = 0 \end{cases}$$

whose simultaneous solution yields the desired values of $m_1$ and $k$. Expanding (31), we have

$$(32) \qquad \begin{cases} \sum_x \bar{y}_x f(x) - m_1 \sum_x xf(x) - k \sum_x f(x) = 0 \\ \sum_x \bar{y}_x xf(x) - m_1 \sum_x x^2 f(x) - k \sum_x xf(x) = 0. \end{cases}$$

Since

$$\sum_x \bar{y}_x f(x) = \sum_x \sum_y yf(x, y) = N\bar{y},$$

and

$$\sum_x \bar{y}_x xf(x) = \sum_x x \sum_y yf(x, y) = \sum_{x,y} xyf(x, y),$$

equation (32) becomes

$$(33) \qquad \begin{cases} N\bar{y} - m_1N\bar{x} - Nk = 0 \\ \sum_{x,y} xyf(x, y) - m_1 \sum_x x^2 f(x) - kN\bar{x} = 0. \end{cases}$$

Solving (33) for $m_1$ and $k$ we find

$$k = \bar{y} - m_1\bar{x}$$

$$m_1 = \frac{\sum_{x,y} xyf(x, y) - N\bar{x}\bar{y}}{\sum_x x^2 f(x) - N\bar{x}^2} = \frac{r\sigma_y}{\sigma_x},$$

[1] Differentiating partially $\sum f(x)(\bar{y}_x - m_1x - k)^2$ with respect to $m$ and $k$ respectively, and setting the results equal to zero, yields equation (31). Instead of differentiating this expression one may expand it, regard the result as a quadratic in both $m$ and $k$, and use the theorem of §3, Chapter VII, to obtain (31).

and the equation of our line becomes

$$\bar{y}_x = r\frac{\sigma_y}{\sigma_x}x + \bar{y} - \frac{\sigma_y}{\sigma_x}r\bar{x},$$

that is

$$\bar{y}_x - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}). \tag{8a}$$

Therefore, the best-fitting line for the means of the columns properly weighted, and the best-fitting line for all the dots are one and the same straight line. But from the point of view of a correlation table, a regression line is to be regarded as the equation from which may be estimated the *average* of all the $y$'s associated with a particular value of $x$. In other words, a prediction in the latter case professes to give only the mean result (Figure 38).

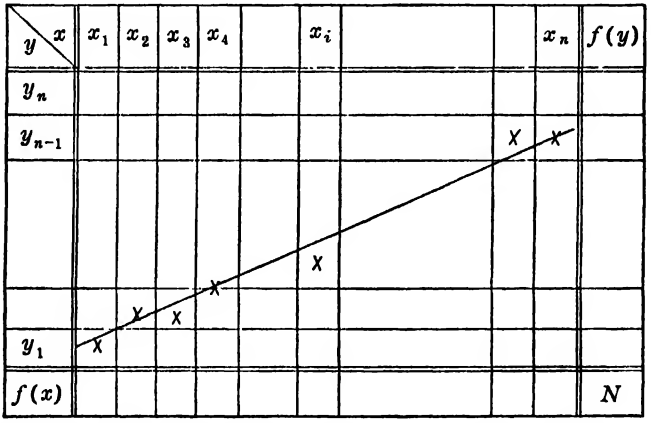



FIG. 38 — THE LINE OF REGRESSION OF $y$ ON $x$ IS THE BEST FITTING LINE FOR THE MEANS OF THE COLUMNS

**16. Applications.** The data of a correlation table are usually regarded as a sample of the much larger class of similar data constituting the universe. A regression equation calculated from a limited but representative sample may give valuable estimates of the average values of $y$ in the universe associated with designated values of $x$.

Let us consider the data of Table 36 on page 203. Suppose a personnel manager in charge of hiring employees of a manufacturing plant has instituted a system of mental tests for applicants, and has gathered these data showing the relationship between the standing

made by applicants on their mental tests and their productive ability when measured according to a certain standard of production after they are hired.

TABLE 36

| $y$ \ $x$ / $v$ \ $u$ | | 22.5 | 27.5 | 32.5 | 37.5 | 42.5 | 47.5 | 52.5 | 57.5 | $f(y)$ = $f(v)$ | $\bar{x}_y$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | | |
| 125 | 4 | | | | | 2 | 3 | 2 | | 7 | 47.5 |
| 115 | 3 | | | 1 | 3 | 1 | 4 | 4 | 4 | 17 | 48.1 |
| 105 | 2 | | | 5 | 7 | 8 | 11 | 8 | 7 | 46 | 45.9 |
| 95 | 1 | | 2 | 1 | 10 | 12 | 9 | 8 | 2 | 44 | 44.0 |
| 85 | 0 | 1 | 3 | 12 | 11 | 7 | 12 | 7 | 1 | 54 | 40.7 |
| 75 | −1 | 2 | 1 | 5 | 6 | 16 | 8 | 5 | | 43 | 41.6 |
| 65 | −2 | 2 | 5 | 5 | 8 | 8 | 6 | 1 | | 35 | 38.0 |
| 55 | −3 | 2 | 3 | 3 | 4 | 1 | 1 | | | 14 | 33.2 |
| $f(x)=f(u)$ | | 7 | 14 | 32 | 49 | 55 | 54 | 35 | 14 | 260 | |
| $\bar{y}_x$ | | 67.9 | 72.1 | 81.9 | 84.8 | 85.7 | 90.9 | 95.6 | 105.0 | | |

Here $x$ represents the grade made on mental test, and $y$ the per cent of standard in production. (See also Table 27.) The means of columns are denoted by $\bar{y}_x$, and the means of rows $\bar{x}_y$.

In order to demonstrate to the company's management the connection between his mental tests and the productivity of the employees he has hired, the personnel manager does the following: (1) Computes the coefficient of correlation between the two series; (2) Shows what the estimated productivity of employees would be whose grades in the mental test fell on the mid-points of the class intervals of the mental test data.

The means of the columns and of the rows are given in the table. In addition, he obtains the following results:

$$\bar{x} = 42.17, \qquad \sigma_y = 17.41, \qquad r = .417,$$

$$\bar{y} = 87.31, \qquad \sigma_x = 8.40, \qquad m_1 = r\frac{\sigma_y}{\sigma_x} \quad .864.$$

Therefore, the line of regression of $y$ on $x$ is

$$\bar{y}_x - 87.31 = .864(x - 42.17)$$

or

$$\bar{y}_x = .864x + 50.88. \tag{34}$$

This is the equation of the line that best fits the points which designate the means of the columns (Figure 39). Hence, for an assigned value of $x$, equation (34) gives the value of $y$ which is the expected mean of the column defined by the assigned value of $x$. The personnel manager is thus prepared to predict the productivity of applicants on the basis of their mental test grades. In other words, the regression equation calculated from the records of those already hired may be used in selecting from future applicants those most likely to succeed.[1]

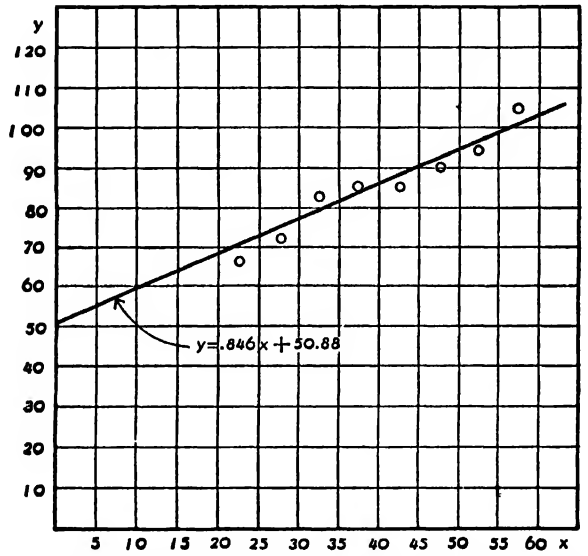



FIG. 39 — MEANS OF COLUMNS AND LINE OF REGRESSION OF $y$ ON $x$ FOR TABLE 36

## Exercises

1. Verify the value of $r$ given for Table 36.
2. Verify the means of the columns given in Table 36.
3. Using equation (34) show what the estimated productivity of employees in the factory referred to above would be whose mental test grades were 22.5, 27.5, etc.
4. For Table 35,
   (a) Find the equations of the regression lines.

[1] The critical reader may doubt if the value $r = .417$ is sufficiently large to warrant much confidence in (34) as a predicting equation. The question of reliability of predictions is discussed later.

(*b*) Locate the axes through the mean of the table and graph the regression lines.

(*c*) Compute $S_y$.

**5.** As in Exercise 4 for the table on page 200.

*Ans.* to (*a*),

$$\bar{y}_x = .069x + 58.3$$
$$\bar{x}_y = 3.36y - 89.4.$$

**17. $S_y$ for a Correlation Table.** For ungrouped data we have defined $S_y$ as a measure of the clustering of the data around the regression line, and have observed that it is called the standard error of estimate. In order to understand what $S_y$ has to do with "estimates" it is necessary first to consider its meaning in a correlation table. Let us denote by $s_{y \cdot x}$ the standard error about the regression line in the array of $y$'s at $x$. Thus we have

$$s_{y \cdot x}^2 = \frac{1}{f(x)} \sum_y ({}_o y - {}_c\bar{y}_x)^2 f(x, y) \tag{35}$$

where ${}_o y$ denotes an observed $y$ value and ${}_c\bar{y}_x$ denotes the value obtained from the regression line for that column. Thus, for the column headed 32.5 in Table 36 we obtain the computed value $\bar{y}_x$ by substituting $x = 32.5$ in (34) whence we find $\bar{y}_x = 78.96$. To evaluate $s_{y \cdot x}^2$ for this column we find the square of the deviation of each of the 32 values of ${}_o y$ from 78.96, add the results and divide by 32. Extracting the square root of the result we find $s_{y \cdot x} = 15.96$. Moving along the regression line suppose we have computed an $s_{y \cdot x}^2$ for each array of $y$'s and averaged the results. It is interesting to learn that this average is $S_y^2$. This is stated more precisely in the following theorem.

**Theorem V.** *The arithmetic mean of the values of $s_{y \cdot x}^2$ for the several columns when each $s_{y \cdot x}^2$ is weighted with the frequency in that column is $S_y^2 = \sigma_y^2(1 - r^2)$.*

*Proof:* Using (35) we have

$$\frac{1}{N} \sum_x f(x) s_{y \cdot x}^2 = \frac{1}{N} \sum_x \sum_y ({}_o y - {}_c\bar{y}_x)^2 f(x, y).$$

Substituting the value given by (8*a*), §15, in the right member of the above identity we have

$$\frac{1}{N} \sum_x \sum_y \left[ y - \left\{ \bar{y} + r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \right\} \right]^2 f(x, y),$$

that is

$$\frac{1}{N}\sum_{x,y}\left\{(y-\bar{y})-r\frac{\sigma_y}{\sigma_x}(x-\bar{x})\right\}^2 f(x,y)$$

which reduces to $\sigma_y^2(1-r^2)$. It is left as an exercise for the student to show this.

For Table 36 we find $S_y = 15.83$. In Figure 40 the parallel lines on either side of the regression line $RR'$ are drawn at a vertical distance of $\pm S_y$ from it. They describe the average limits of scatter above and below the regression line.

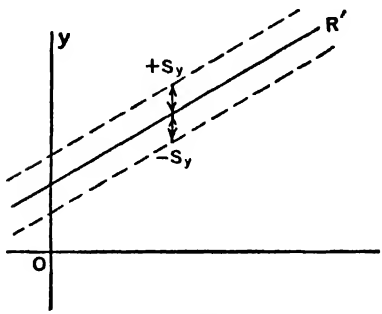



FIG. 40

To connect $S_y$ with the reliability of predictions it is necessary to introduce the concept of a correlation surface. Indeed, a knowledge of the fundamental properties of a correlation surface is desirable for a wider outlook on correlation theory in general.

**18. Normal Correlation Surface.** A correlation table may be idealized into a surface in somewhat the same way that a histogram is idealized into a frequency curve. The concept of a surface relates to the universe from which the observed data of the table may be regarded as a sample. Let the dimensions of the cells of a table be $\Delta x$ and $\Delta y$, and suppose columns are erected upon these cells with altitudes proportional to the frequencies in the cells. The result is a sort of solid histogram. Then as $\Delta x \to 0$, $\Delta y \to 0$, $N \to \infty$, the tops of the columns approach as a limit a smooth surface which is called a correlation surface. Our discussion will be confined to the case where we may assume that this limit is a normal correlation surface. In discussing this surface it is convenient to let $x$ and $y$ represent deviations from the respective means and to let $z = f(x, y)$ denote the frequency function representing the surface. Such a surface is shown in Figure 41.

Any section of this surface parallel to the $yz$-plane is a normal curve and represents the distribution in a column at $x$. Similarly any section parallel to the $xz$-plane representing a row is a normal curve. The frequency in a cell is measured by that portion of the volume under the surface which lies over that cell. All those cells

in which the frequency is a fixed value lie on an ellipse. That is, if contour lines are drawn on the surface joining the points of equal height above the base they will be ellipses. In other words, sections of the surface parallel to the $xy$-plane are ellipses.

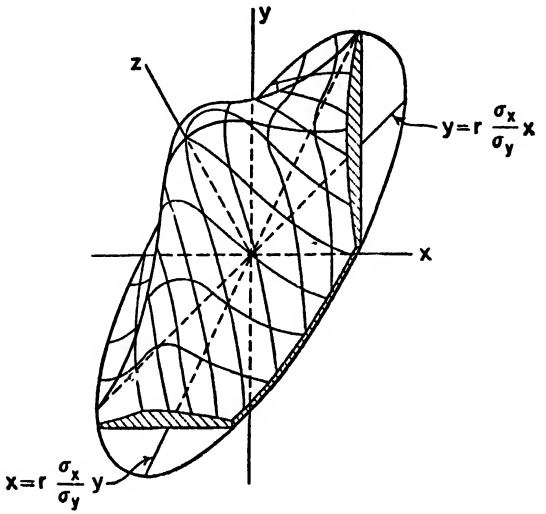



FIG. 41 — FREQUENCY SURFACE FOR CORRELATED VARIABLES

We will digress here for a brief discussion of an ellipse. We may think of an ellipse as a transitional figure between a circle and a straight line, as the circle flattens out. That is to say, the limiting form of an ellipse is a circle at one extreme of the flattening process and a straight line segment at the other extreme. The degree of flatness is called the *eccentricity* of the ellipse, and it is proved in analytic geometry that the eccentricity varies from zero in the case of a circle to unity when the ellipse degenerates into a line. All ellipses having the same eccentricity whatever their size have the same relative proportions and are therefore similar in form.

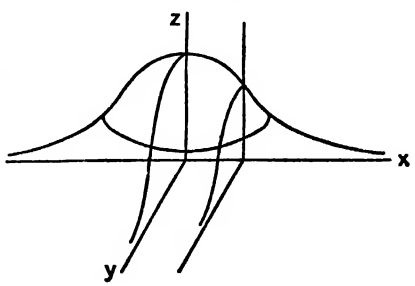



FIG. 42

The eccentricity of the elliptical contours of different normal cor-

relation surfaces varies with the amount of correlation existing in the corresponding universe. A surface with narrow elliptical contours represents a universe in which there is high correlation, whereas if the variables are completely independent in the probability sense the contour lines are circles when the variables are expressed in standard units. If the variables are not expressed in standard units (and $r = 0$) then the contour lines may be ellipses but their major and minor axes will coincide with the $x$- and $y$-axes as in Figure 42. When $r \neq 0$ the axes of the ellipses make an angle with the $xy$-axes, their major axis cuts quadrants I and III in the $xy$-plane if $r > 0$ (as in Figure 41) and quadrants II and IV if $r < 0$.

**19. Properties of Normal Bivariate Surface.** The equation of a normal correlation surface is given by

$$f(x, y) = Ke^{-P} \tag{36}$$

where

$$P = \frac{1}{2(1 - r^2)}\left\{\frac{x^2}{\sigma_x{}^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y{}^2}\right\},$$

$K = N \div (2\pi\sigma_x\sigma_y\sqrt{1 - r^2})$, and $x$ and $y$ represent the correlated variables referred to their respective means as origin.

By means of (36) an observed distribution may be fitted with the appropriate normal surface assuming that the sample might reasonably have come from such a universe. This is accomplished by replacing $\sigma_x$, $\sigma_y$, $r$, and $N$ in (36) by the corresponding statistics calculated from the sample and taking the origin at the mean of the table. Let us assume that an observed distribution has been graduated by such a surface and the theoretical cell frequencies obtained. The surface extends to infinity in the $xy$-plane but contour ellipses can be obtained which will enclose any desired percentage of the given frequency when these ellipses are projected orthogonally onto the $xy$-plane. They are all concentric, similar, and similarly placed. Figure 43 represents such an ellipse, say the smallest one necessary to enclose all the given cells. The systems of perpendicular chords represent the columns and rows of the table.

The graduated frequencies for each column are normal distributions whose means lie on the regression line of $y$ on $x$ and whose standard deviations are in each case given by $S_y = \sigma_y(1 - r^2)^{1/2}$. To state the same thing in a slightly different way, an array of $y$'s corresponding to a fixed value $x_1$ of $x$ is a normal distribution whose

mean deviates from $\bar{y}$ by $r(\sigma_y/\sigma_x)x_1$ and whose standard deviation is $S_y = \sigma_y(1 - r^2)^{1/2}$ which is independent of $x_1$ and therefore is the same for all such arrays. Similarly an array of $x$'s corresponding to a particular value $y_1$ of $y$ is a normal distribution with a mean which deviates from $\bar{x}$ by $r(\sigma_x/\sigma_y)y_1$, and a standard deviation of $S_x = \sigma_x(1 - r^2)^{1/2}$ which is independent of $y_1$ and therefore is the same for all such arrays. A careful study of Figure 41 will help in understanding what is meant by these statements.

When the means $\bar{y}_x$ of the columns fall exactly on the regression line, $s_{y \cdot x}$ becomes the standard deviation of a column and is therefore the same as $S_y$. Theorem V states that $S_y^2$ is an average of the values of $s_{y \cdot x}^2$ but when all the quantities being averaged have the same value, as they do in the ideal case of the normal surface, their (mean) average is that value. When the standard deviations of the columns are equal, the regression system of $y$ on $x$ is called a *homoscedastic* system. In a universe where they are not equal the system is said to be *heteroscedastic*. For a homoscedastic system with linear regression, $S_y = \sigma_y(1 - r^2)^{1/2}$ is the standard deviation of each array of $y$'s.

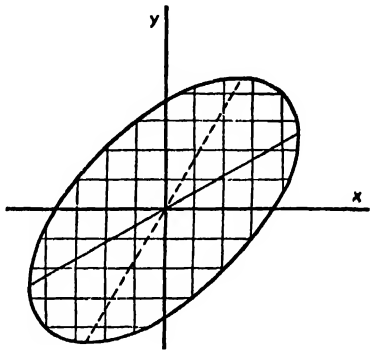


FIG. 43

**20. Reliability of Predictions.** In using a regression equation to make predictions we are naturally interested in the degree of confidence to be expected in the predictions thus made. The use of $S_y$ in this connection is based upon the properties of the normal correlation surface.

Let us imagine the universe of which Table 36 is a sample and assume that it may be described by a normal surface. Confining our attention to a section parallel to the $yz$-plane in Figure 41 we know that an $x$ array of $y$'s is distributed normally about a value of $y$ determined by a designated value of $x$ in the regression equation of $y$ on $x$. That is, the mean of this normal distribution is the predicted value of $y$ and its standard deviation is $S_y$. The percentage distribution of such an array is the same as that given in Figure 23 of Chapter VI, if $S_y$ is taken as the unit of measurement along the horizontal axis. But an estimate of $S_y$ is its value cal-

culated from the sample. Moreover, for an observed distribution, we have seen that $S_y$ is the average standard deviation of the several columns and therefore it may reasonably be taken as an approximation to the theoretical $S_y$ which in the universe is the same for all the columns. We also take the calculated regression equation as an approximation to the theoretical.

By measuring deviations from the predicted value in terms of $S_y$ in the same way that $\sigma$ is used as a unit in measuring deviations from the mean, we may then enter a normal probability scale for the probability of a deviation involving multiples of $S_y$. According to this scale the probability $P_y$ is about .68 for a deviation of $\pm S_y$ from the predicted value, and the chances are even for a deviation of $.6745S_y$ on either side of the predicted value.

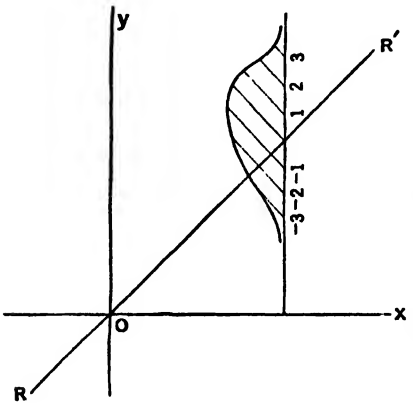



FIG. 44 — REPRESENTING AN $x$ ARRAY OF $y$'S AND DEVIATIONS OF $\pm S_y$ FROM A PREDICTED VALUE OF $y$

For Table 36 we have found $S_y = 15.83$ and for an applicant making $x = 32.5$ on the mental test we have predicted $y = 78.96$. Therefore the chances are about 68 in 100 that his percentage of productivity will be between $78.96 - 15.83$ and $78.96 + 15.83$, that is, between 63.13 and 94.79. In other words, the probability is about .68 that the predicted value will not be in error by more than 15.83.

To summarize, in a normal bivariate universe each array is a normal distribution and therefore its mean coincides with its mode. Since regression is linear, a value predicted from the regression equation of $y$ on $x$ is the mean value of $y$ for a designated value of $x$. Then, $P_y = \int_{-t}^{t} \phi(t)\, dt$ is the probability for a deviation from the predicted value of $\bar{y}_x$ as small as $|t|$ where $t$ is expressed in units of the standard error $S_y$ of a column. Thus,

$$t = \frac{y - \bar{y}_x}{S_y}.$$

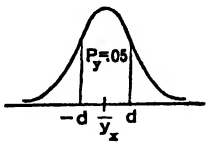
Then $1 - P_y$ is the probability for a deviation as large as $|t|$. Similarly, when dealing with the regression line of $x$ on $y$, $P_x = \int_{-t}^{t}$ is the probability for a deviation from the predicted value $\bar{x}_y$ as small as $|t|$, where now $t = (x - \bar{x}_y)/S_x$.

## Exercises

1. Refer to problem 4, §4. Assume that the data given there are obtained from a correlation table which is a representative sample from a normal bivariate universe describing the heights and weights of senior men students in colleges and universities of the United States. Then a value predicted from the regression equation of $y$ on $x$ will give the mean of the "column" at $x$. Similarly, for an assigned $y$, the corresponding $x$ in the regression equation of $x$ on $y$ will be the mean of the "row" at $y$. Under this assumption, determine the probability that Doe's height is outside the interval 65.75 — 77.75 inches. What are the chances that Roe will be between 100.8 and 122.4 pounds in weight?
   *Ans.* $1 - P_y = .0027$, $P_x = .5$ (approximately).
2. Discuss the reliability of the predictions which you made in Exercise 3, §16.
   *Outline of Solution.* Suppose a reliability level of $P_y = .5$ is desired. Making the necessary assumptions, this allows a deviation of $t = \pm.6745$. Since $S_y = 15.83$ we have

$$\pm t = \frac{d}{15.83}$$

   where $d = y - \bar{y}_x$. That is, $y = \bar{y}_x \pm ?$ For $x = 37.5$, $y = ? \pm ?$ So the probability is .5 that the standard of production will be between what limits for a person making $x = 37.5$ on the mental test? The problem is analogous for any other designated value of $P_y$ and for other assigned values of $x$.




3. Consider the surface represented by (36). Prove that a section of the surface parallel to the $yz$ coördinate plane is a normal curve with its mean on the regression line of $y$ on $x$ and with variance $S_y^2 = \sigma_y^2(1 - r^2)$.
   *Outline of Solution.* Write (36) in the form

$$f = Ke^{-P}, \tag{a}$$

   where $P = (u^2 - 2ruv + v^2)/2(1 - r^2)$, $u = x/\sigma_x$, $v = y/\sigma_y$, $\bar{x} = 0 = \bar{y}$. The *trace* of the surface in the plane $u = u_1$ is determined by substituting $u_1$ for $u$ in (*a*). This substitution yields the result

$$f = Ce^{-T} \tag{b}$$

where $T = (v - ru_1)^2/2(1 - r^2)$, $C = Ke^{-u_1^2/2}$. Upon returning to $(x, y)$ coördinates, $(b)$ becomes

$$(c) \qquad f = Ce^{-h^2(y-m)^2},$$

where $m = rx_1\sigma_y/\sigma_x$, $h^2 = 1/(2S_y^2)$, $S_y^2 = \sigma_y^2(1 - r^2)$.

**21. Non-Linear Regression. Correlation Ratio.** We have seen that the regression systems of a normal correlation surface are linear. In a correlation table which is a representative sample from a normal bivariate universe the means of the arrays would lie approximately on straight lines. But in correlation tables which are samples of other types of universes, regression might not be linear. Moreover, one of the regression curves might be strictly linear and the other non-linear. The following numerical example illustrates the latter possibility.

| $y$ \ $x$ | 0 | 1 | 2 | 3 | $f(y)$ |
|---|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 2 | 2 |
| 1 | 0 | 1 | 1 | 2 | 4 |
| 0 | 1 | 1 | 0 | 0 | 2 |
| $f(x)$ | 1 | 2 | 1 | 4 | 8 |

In this example, the regression of $y$ on $x$ is linear whereas that of $x$ on $y$ is non-linear.

When the means of the columns (or of the rows) do not lie approximately on a straight line, the use of $r$ may be misleading because $r = 0$ indicates absence of linear correlation only and not necessarily absence of correlation in general.

One of the best treatments of this situation is that given in the *Carus Monograph on Mathematical Statistics*, which will be reproduced substantially here.

In introducing a *correlation ratio*, $\eta_{yx}$, (eta) of $y$ on $x$, as an appropriate measure of correlation to take the place of the correlation coefficient in such a situation, we may get suggestions as to what is appropriate by solving for $r$ in (10). This gives

$$(37) \qquad r^2 = 1 - \frac{S_y^2}{\sigma_y^2},$$

where we may recall that $S_y^2$ is the mean square of deviations from the line of

regression. Then

$$r = \pm\left[1 - \frac{S_y^2}{\sigma_y^2}\right]^{1/2}.$$

This formula could be used appropriately as a definition of $r$ in place of our definition in (1), and its examination may throw further light on the significance of $r$. When $S_y = 0$, the formula gives $r = \pm 1$ and, as we have seen earlier, all the dots of the scatter diagram must then fall exactly on the line of regression. When $S_y = \sigma_y$, the formula gives $r = 0$, and the regression line is in this case of no aid in predicting the value of $y$ from assigned values of $x$. In the formula $r^2 = 1 - S_y^2/\sigma_y^2$ it is important to keep in mind that the mean square deviation $S_y^2$ is from the line of regression. Next, let $S_y'^2$ be the corresponding mean square of deviations from the means of columns. Then $S_y'^2 = S_y^2$ when the regression is strictly linear, but $S_y'^2 \neq S_y^2$ when the regression is non-linear. This fact suggests the use of a formula closely related to $[1 - S_y^2/\sigma_y^2]^{1/2}$ for a measure of non-linear regression by replacing $S_y$ by $S_y'$. We than write

$$\eta_{yx}^2 = 1 - \frac{S_y'^2}{\sigma_y^2} \tag{38}$$

where $\eta_{yx}$ is the correlation ratio of $y$ on $x$, and $S_y'^2$ is the mean square of deviations from the means of the columns whether these means are near to or far from the line of regression.

In general, we may say that the correlation ratio of $y$ on $x$ is a measure of the clustering of dots about the means of columns.

An analogous discussion for the rows obviously leads to

$$\eta_{xy}^2 = 1 - \frac{S_x'^2}{\sigma_x^2}$$

giving $\eta_{xy}^2$, the square of the correlation ratio of $x$ on $y$.

That $\eta_{yx}^2 \leqq 1$ and that the equality holds only when all the dots in each column are at the mean of the column follows at once from (38).

That $\eta_{yx}^2 \geqq r^2$ may be shown by recalling the meanings of $S_y^2$ in (37) and of $S_y'^2$ in (38). A mean square of deviations in each column is a minimum when the deviations are taken from the mean of the array. Hence, the $S_y'^2$ in (38) must be equal to or less than $S_y^2$ in (37) for the same data, since the deviations in (37) are measured from the line of regression. Hence, we have shown that

$$1 \geqq \eta_{yx}^2 \geqq r^2.$$

Moreover, when the regression of $y$ on $x$ is linear, $\eta_{yx}^2 - r^2$ found from the sample differs from zero by an amount not greater than the fluctuations due to random sampling. Hence, $\eta_{yx}^2 - r^2$ becomes a criterion for testing the linearity of the regression of $y$ on $x$.

For computational purposes, it is desirable to express the correlation ratios in a form involving the standard deviations of the means of arrays. For this purpose, let $\bar{y}_x$ be the mean of any column of $y$'s and $\sigma_{\bar{y}_x}$ the standard deviation of the means of columns when the square $(\bar{y}_x - \bar{y})^2$ of each deviation is weighted with the number $f(x)$ in the column. Then it follows that

$$\eta_{yx}^2 = \frac{\sigma_y^2 - S_y'^2}{\sigma_y^2} = \frac{\sigma_{\bar{y}_x}^2}{\sigma_y^2}. \tag{39}$$

That is, the correlation ratio of $y$ on $x$ is the ratio of the standard deviation of the means of columns to the standard deviation of all $y$'s.[1]

To prove (39) we must show that $\sigma_y^2 - S_y'^2 = \sigma_{\bar{y}_x}^2$. We begin by observing that the concentration of the dots in a column about their mean may be measured in terms of their standard deviation. Let $\sigma_{y \cdot x}$ denote the standard deviation of the $y$'s in the column at $x$. That is,

$$\sigma_{y \cdot x}^2 = \frac{1}{f(x)} \sum_y (y - \bar{y}_x)^2 f(x, y). \tag{40}$$

Now, the concentration of the dots in the entire table about the means of the columns may be measured by finding the mean value of all such expressions $\sigma_{y \cdot x}^2$ for all the columns of the table. But since there are more points in some columns than in others, it will be desirable to weight the $\sigma_{y \cdot x}^2$ for each column by multiplying it by the number of points or dots in the column. It is this weighted mean value of the $\sigma_{y \cdot x}^2$'s which we have denoted by $S_y'^2$. That is,

$$S_y'^2 = \frac{1}{N} \sum_x f(x) \sigma_{y \cdot x}^2. \tag{41}$$

In order to verify (39) we must now show that

$$\sigma_y^2 = S_y'^2 + \sigma_{\bar{y}_x}^2.$$

Adapting (14) of §9, Chapter V, to the notation of this chapter, we have

$$N\sigma_y^2 = \sum_x f(x) \sigma_{y \cdot x}^2 + \sum_x f(x) (\bar{y}_x - \bar{y})^2. \tag{42}$$

This follows from the fact that $N$ is composed of the several sub-distributions $f(x)$ in the columns, and $\sigma_{y \cdot x}$ is the standard deviation of a column about its mean $\bar{y}_x$. It is obvious that

$$\frac{1}{N} \sum_x f(x) (\bar{y}_x - \bar{y})^2$$

gives the variance $\sigma_{\bar{y}_x}^2$ of the means of the columns. The above expression (42) then becomes

$$N\sigma_y^2 = NS_y'^2 + N\sigma_{\bar{y}_x}^2,$$

which reduces to $\sigma_y^2 - S_y'^2 = \sigma_{\bar{y}_x}^2$, and hence we obtain (39).

**22. Computation of $\eta^2$.** It should be instructive to compute $\eta_{yx}^2$ for Table 36, by both relations (38) and (39).

[1] Rietz, *Carus Monograph on Mathematical Statistics*, p. 89 *et seq.*

For (38) we have the following:

$$\eta_{yx}^2 = 1 - \frac{S_y'^2}{\sigma_y^2}, \qquad S_y'^2 = \frac{1}{N}\sum_x f(x)\sigma_{y\cdot x}^2,$$

$$\sigma_{y\cdot x}^2 = \frac{1}{f(x)}\sum_y (y - \bar{y}_x)^2 f(x, y).$$

| $\sigma_{y\cdot x}^2$ | $f(x)$ |
|---|---|
| 106.12[1] | 7 |
| 191.83 | 14 |
| 246.48 | 32 |
| 283.63 | 49 |
| 257.65 | 55 |
| 294.51 | 54 |
| 222.53 | 35 |
| 71.43 | 14 |

$$S_y'^2 = 246.45$$

$$\sigma_y^2 = (17.41)^2 = 303.11$$

$$\eta_{yx}^2 = 1 - \frac{246.45}{303.11}$$

$$= .1869.$$

For (39) we have the following:

$$\eta_{yx}^2 = \frac{\sigma_{\bar{y}_x}^2}{\sigma_y^2}, \qquad \sigma_{\bar{y}_x}^2 = \frac{1}{N}\sum_x (\bar{y}_x - \bar{y})^2 f(x),$$

$$\bar{y} = 87.31.$$

| $\bar{y}_x$ | $f(x)$ |
|---|---|
| 67.86 | 7 |
| 72.14 | 14 |
| 81.87 | 32 |
| 84.80 | 49 |
| 85.73 | 55 |
| 90.92 | 54 |
| 95.57 | 35 |
| 105.00 | 14 |

$$\sigma_{\bar{y}_x}^2 = 56.66 \text{ (see Exercise 3, p. 97).}$$

$$\eta_{yx}^2 = \frac{56.66}{303.11}$$

$$= .1869.$$

[1] See Table 27 and Table 16.

In verifying (39) for this example we have $\sigma_y^2 - S_y'^2 = 303.11 - 246.45 = 56.66$ and $\sigma_{\bar{y}_x}^2 = 56.66$.

The above illustrations are useful in giving an understanding of the meaning of $\eta_{yx}^2$. However, for computational purposes, another formula may be derived which involves less labor than either (38) or (39). In fact, the computation of a correlation ratio may be very conveniently performed by an easy extension of a correlation table. The derivation of the appropriate formula will now be given.

The standard deviation ($\sigma_{\bar{y}_x}$) of the means of the columns may be expressed in the $(u, v)$ units by the relation $\sigma_{\bar{y}_x}^2 = k^2\sigma_{\bar{v}_u}^2$

where
$$\sigma_{\bar{v}_u}^2 = \frac{1}{N}\sum_u f(u)\bar{v}_u^2 - \bar{v}^2$$

which is the definition of the standard deviation of the variable $\bar{v}_u$. This is apparent if we observe that the mean for the whole table in the $v$-direction ($\bar{v}$) is the mean of the quantities $\bar{v}_u$ for the several columns.[1]

Since
$$\bar{v}_u = \frac{1}{f(u)}\sum_v vf(u, v) = \frac{V}{f(u)},$$

we have
$$\sigma_{\bar{v}_u}^2 = \frac{1}{N}\sum_u \frac{V^2}{f(u)} - \bar{v}^2.$$

Recalling that $\sigma_y^2 = k^2\sigma_v^2$, we have

$$\eta_{yx}^2 = \frac{\sigma_{\bar{y}_x}^2}{\sigma_y^2} = \frac{k^2\left\{\frac{1}{N}\sum_u \frac{V^2}{f(u)} - \bar{v}^2\right\}}{k^2\sigma_v^2};$$

that is,

$$\eta_{yx}^2 = \frac{1}{\sigma_v^2}\left\{\frac{1}{N}\sum_u \frac{V^2}{f(u)} - \bar{v}^2\right\}. \tag{43}$$

An analogous discussion for the rows of $x$'s leads to

$$\eta_{xy}^2 = \frac{1}{\sigma_u^2}\left\{\frac{1}{N}\sum_v \frac{U^2}{f(v)} - \bar{u}^2\right\}, \tag{44}$$

giving the square of the correlation ratio of $x$ on $y$.

[1] $\frac{1}{N}\sum_u f(u)\bar{v}_u = \frac{1}{N}\sum_u f(u)\frac{1}{f(u)}\sum_v vf(u, v) = \frac{1}{N}\sum_u\sum_v vf(u, v) = \frac{1}{N}\sum_{u,v} vf(u, v) = \bar{v}.$

*Example.* Find $\eta_{yx}^2$ for Table 35. *Solution:* Referring to this table and using (43) we obtain the following results:

| $V^2$ | 49 | 9 | 9 | 49 | 900 | 121 | 169 | Sum |
|---|---|---|---|---|---|---|---|---|
| $V^2/f(u)$ | 3.78 | .75 | .47 | 2.45 | 37.50 | 20.17 | 28.17 | 93.29 |

$$\bar{v}^2 = .2916, \qquad \sigma_v^2 = 1.8084, \qquad N = 100.$$

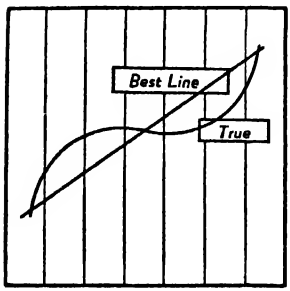
$$\eta_{yx}^2 = \frac{1}{1.8084}\left[\frac{1}{100}(93.29) - .2916\right].$$

$$\eta_{yx}^2 = .3546.$$

It may be well to mention that the value of $\eta$ is not independent of the classification of the data. As the class intervals become narrower, $\eta$ approaches unity. This may be understood from (38). If the grouping were so fine that only one item appeared in each column, then it would constitute the mean of that column. In this case $S_y'$ would be zero and $\eta$ would therefore be unity. On the other hand, a very coarse grouping tends to make the value of $\eta$ approach $r$. "Student" has given a formula for *The Correction to be Made in the Correlation Ratio for Grouping* in Biometrika, vol. IX, pp. 316–320.

**23. Further Discussion. Test for Linearity of Regression.** Let us consider the totality of mean points $(x, \bar{y}_x)$ of the columns and think of a curve connecting them. Of course, for a table of observed data, it is possible to draw many such curves. In order to show clearly why a comparison of $\eta^2$ and $r^2$ is the basis of a test for linearity of regression, it will be necessary to consider a theoretical table in which there is only one such curve. When we speak of *the* regression curve we are thinking, not of the given table in which the dimensions of the cells are $h$ and $k$, but of an ideal table in which there is an infinity of cells of zero dimensions. To put it another way, consider a sample of $N$ pairs of values $(x_i, y_i)$ from which a correlation table is made with cells whose dimensions are $h$ and $k$. If parallelepipeds are erected on the cells with heights proportional to the frequencies, the result is a solid histogram bounded by a broken surface. As $h \to 0$, $k \to 0$, and $N \to \infty$, this histogram will approach some solid, bounded by a smooth surface. An example of such a surface is the normal correlation surface. In such an ideal table, it is possible to have but one curve connecting the means of the columns. This curve is sometimes styled the *true* regression curve of $y$ on $x$. In an

analogous way for the means of the rows there would be a *true* regression curve of $x$ on $y$. It is one of these curves that we have in mind when we speak of " the regression curve " or " the regression." For a normal bivariate universe (represented by a normal correlation surface), regression is linear. But for other types of bivariate universes (which might be represented by skew surfaces), it is conceivable that regression might be parabolic or exponential or some other type of curve. In such types, regression is said to be non-linear. The curve which is chosen to approximate the true regression curve must not be confused with the true regression curve. The latter notion relates to the ideal universe from which the data at hand are a sample. It is defined as the locus of the mean points of the columns of the theoretical table. When we fit a curve to the means of the columns of an observed table, this regression curve is merely an approximation to the ideal set up in the definition. Similar statements may be made about the regression of $x$ on $y$.




We will now recapitulate the expressions used in the comparative analysis of $r^2$ and $\eta_{yx}^2$ for an observed table.

$$(45)\qquad \begin{cases} \sigma_{y\cdot x}^2 = \dfrac{1}{f(x)}\sum_y (y - {}_c\bar{y}_x)^2 f(x, y) \\ S_y'^2 = \dfrac{1}{N}\sum_x \sigma_{y\cdot x}^2 f(x) \\ \eta_{yx}^2 = 1 - \dfrac{S_y'^2}{\sigma_y^2} = \dfrac{\sigma_{\bar{y}_x}^2}{\sigma_y^2} \end{cases}$$

$$(46)\qquad \begin{cases} s_{y\cdot x}^2 = \dfrac{1}{f(x)}\sum_y (y - {}_c\bar{y}_x)^2 f(x, y) \\ S_y^2 = \dfrac{1}{N}\sum_x s_{y\cdot x}^2 f(x) \\ r^2 = 1 - \dfrac{S_y^2}{\sigma_y^2}. \end{cases}$$

Recall that $\sigma_{y\cdot x}^2$ is defined as the variance in a column and therefore as the square of the standard error about the regression curve, what-

ever it may be, which goes through the means of the columns. $S_y'^2$ is an average of the $\sigma_{y\cdot x}^2$ values, and $\eta_{yx}^2$ is defined in terms of $S_y'$. Correspondingly, $s_{y\cdot x}^2$ is the square of the standard error in a column about the *line* which best fits the means of the columns. $S_y^2$ is an average of the $s_{y\cdot x}^2$ values, and $r^2$ is defined in terms of $S_y$. If regression is linear, the means of the columns will fall on the "best-fitting line" and $\sigma_{y\cdot x}^2$ becomes the same as $s_{y\cdot x}^2$. Then $S_y'^2 = S_y^2$, and hence $\eta_{yx}^2 = r^2$.

It is interesting to observe that $\sigma_{y\cdot x}^2$ is the second moment about the mean, for an array of $y$'s, *i.e.*, for a column. In the notation of moments it could be denoted by $\mu_{2:y\cdot x}$. In this notation, $s_{y\cdot x}^2$ could be denoted by $\nu_{2:y\cdot x}$, being the second moment in an array of $y$'s about a point other than its mean. Since $\mu_2 \leq \nu_2$, it follows that $\sigma_{y\cdot x}^2 \leq s_{y\cdot x}^2$. Therefore $S_y'^2 \leq S_y^2$ and $\eta_{yx}^2 \geq r^2$. If each $y$ value of a column is at the mean of that column then it is obvious that $\sigma_{y\cdot x}^2$ will be zero. In this case, $S_y' = 0$, and $\eta_{yx}^2 = 1$. On the other hand, for any column, the contribution of $\sigma_{y\cdot x}^2 f(x)$ to $S_y'^2$ cannot exceed its contribution to $\sigma_y^2$. Taking the weighted mean of the respective contributions over all the columns, we have $S_y'^2 \leq \sigma_y^2$ and hence

$$\eta_{yx}^2 \leq 1.$$

Writing (38) in the form

$$S_y' = \sigma_y(1 - \eta_{yx}^2)^{1/2}$$

we see that $S_y'$ is a measure of dispersion about the regression curve (which is the locus of the means) corresponding to $S_y = \sigma_y(1 - r^2)^{1/2}$ which is the standard error about the "best" *line*. If $r^2 = 1$, then $y$ is related to $x$ by a linear function. If $\eta_{yx}^2 = 1$, it follows that $y$ is a single-valued function of $x$. On the other hand, if $r^2 = 0$, it does not necessarily follow that there is no relation[1] between $y$ and $x$. If $\eta_{yx}^2 = 0$ then $r^2 = 0$, but if $r^2 = 0$ it does not necessarily follow that $\eta_{yx}^2 = 0$.

In the ideal table, regression of $y$ on $x$ is linear if and only if $\eta_{yx}^2 - r^2 = 0$. But in the case of an observed table, allowance must be made for sampling fluctuations. A corresponding analysis could be made for $r^2$ and $\eta_{xy}^2$, and $\eta_{xy}^2 - r^2$ computed from the sample should

[1] See H. L. Rietz, "On Functional Relations for which the Coefficient of Correlation is Zero." *Journal American Statistical Association*, vol. 16, 1919, pp. 472–76.

differ from zero by an amount not greater than the fluctuations due to chance, if regression of $x$ on $y$ is linear. The question naturally arises, what discrepancy between the computed values of $\eta^2$ and $r^2$ may be tolerated before we conclude that regression is non-linear? This problem has been investigated, and Blakeman [1] has proposed a testing formula. If certain assumptions are made, a simple though approximate test may be deduced from Blakeman's formula. According to this approximate test if

$$N(\eta^2 - r^2) < 11.4 \tag{47}$$

then linear regression may be assumed. Since there are two $\eta^2$'s there are two tests. It is possible for one of the regression curves to be linear and the other not.

Evaluating (47) for Table 35 we obtain $100\,[.3546 - (.58)^2] = 1.82$, so the regression of $y$ on $x$ may be assumed to be linear.

R. A. Fisher has shown that the Blakeman test is not very reliable. One can easily construct an example for which regression is obviously non-linear yet which satisfies the criterion (47). Consider the following table:

| $y$ \ $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 3 | 0 | 0 | 1 | 0 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |

Here, $N = 5$, $\sum xy = 27$, $\bar{x} = 3$, $\bar{y} = 9/5$. From (3), therefore, $r = 0$. From (40) and (41), $S_y' = 0$ and $\eta_{yx} = 1$. Applying (47), Blakeman's test yields a verdict of linear regression of $y$ on $x$. It appears that Blakeman's criterion is of doubtful utility. A more efficient method of testing linearity of regression is given in Part II.

## Exercises

1. Using (43) and (44) find $\eta_{yz}^2$ and $\eta_{zy}^2$ for the table referred to in Exercise 4, page 221. Apply the test (47) and state your opinion about the linearity of regressions.

[1] See *Handbook of Mathematical Statistics*, Rietz and others, p. 131.

**2.** In the following table, $x$ = Interest Rates, 4–6 months Commercial Paper; $y$ = Total Bills Discounted by Federal Reserve Banks (1923–1932). Find $r$ and $\eta_{yx}^2$. Form an opinion about linearity of regression of $y$ on $x$. (Data from *Elements of Statistics*, Davis and Nelson, page 288.)

| *Class Marks* $y$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 | | | | | | | 1 | 6 | 6 | 6 |
| 6 | | | | | 1 | 2 | 3 | 4 | | |
| 5 | | | | | 1 | 3 | 1 | 2 | | |
| 4 | | | | 2 | | 9 | 4 | 1 | | |
| 3 | | 1 | 2 | 1 | 4 | 9 | 4 | | | |
| 2 | | 1 | | | 11 | 5 | 1 | | | |
| 1 | 4 | | 2 | 3 | 3 | 1 | | | | |
| 0 | 2 | 3 | 3 | 5 | 3 | | | | | |
| *Class Marks* $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**3.** In §44, *Statistical Methods for Research Workers*, R. A. Fisher writes: "The sum of the squares of the deviations of all the values of $y$ from their general mean may be broken up into two parts, one representing the sum of the squares of the deviations of the means of the arrays from the general mean, each multiplied by the number in the array, while the second is the sum of the squares of the deviations of each observation from the mean of the array in which it occurs." [Compare with our (14$a$) of Chapter V.]

Prove Fisher's statement. *Hint.* In symbols, you are to prove that

$$V = v_1 + v_2$$

where

$$V = \sum_{x,\,y} (y - \bar{y})^2 f(x, y)$$

$$v_1 = \sum_{x} (\bar{y}_x - \bar{y})^2 f(x)$$

$$v_2 = \sum_{x,\,y} (y - \bar{y}_x)^2 f(x, y).$$

**4.** Prove that $\eta_{yx}^2$ is the ratio between $v_1$ and $V$ as defined in Exercise 3.

5. The mortality experience during the early years of an insurance company presents an interesting study in correlation. The following table shows for male lives the correlation between the ages ($x$) of the insured at issue of policy and his age ($y$) at death. Data of the Midland Life Insurance Company,[1] 1906–1924.

| $y$ \ $x$ | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | $f(y)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 70 | | | | | | | | | 1 | 2 | 3 |
| 65 | | | | | | | | 4 | 9 | 3 | 16 |
| 60 | | | | | | | 6 | 5 | 7 | 1 | 19 |
| 55 | | | | 1 | | 2 | 12 | 20 | 4 | | 39 |
| 50 | | | | | 2 | 13 | 13 | 8 | | | 36 |
| 45 | | | | 1 | 12 | 12 | 8 | | | | 33 |
| 40 | | | 3 | 13 | 19 | 12 | | | | | 47 |
| 35 | | 1 | 8 | 14 | 14 | | | | | | 37 |
| 30 | | 5 | 10 | 7 | | | | | | | 22 |
| 25 | | 11 | 10 | | | | | | | | 21 |
| 20 | 6 | 4 | | | | | | | | | 10 |
| $f(x)$ | 6 | 21 | 31 | 36 | 47 | 39 | 39 | 37 | 21 | 6 | 283 |

Find $r$, the two $\eta^2$'s, and the equations of the lines of regression.

**24. Correlation from Ranks.** Before defining *rank* we will find the variance of the difference, $z$, between corresponding values of two variables. Let $x$ and $y$ denote corresponding values of two series each consisting of $N$ variates. Form a third series $z$ where $z_i = x_i - y_i$. Then the mean of $z$ is given by $\bar{z} = \bar{x} - \bar{y}$ and the standard deviation of $z$ is, by definition,

$$\sigma_z^2 = \frac{1}{N}\sum z^2 - \bar{z}^2.$$

[1] From a paper *On Certain Applications of Mathematical Statistics to Actuarial Data* in The Record, American Institute of Actuaries, vol. XIII, Part II, No. 28, November, 1924.

Replacing $z$ by its equal $x - y$, we have

$$\sigma_z^2 = \frac{1}{N}\sum(x^2 - 2xy + y^2) - \bar{x}^2 - \bar{y}^2 + 2\bar{x}\bar{y}$$

$$= \left\{\frac{1}{N}\sum x^2 - \bar{x}^2\right\} - 2\left\{\frac{1}{N}\sum xy - \bar{x}\bar{y}\right\} + \left\{\frac{1}{N}\sum y^2 - \bar{y}^2\right\}.$$

Whence

$$\sigma_z^2 = \sigma_x^2 - 2r\sigma_x\sigma_y + \sigma_y^2. \tag{48}$$

If the variables $x$ and $y$ are uncorrelated, we have as a special case

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2.$$

Solving (48) for $r$, we obtain

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x\sigma_y}. \tag{49}$$

This is another expression for the correlation coefficient and involves standard deviations only. In particular, it may be used to advantage when $x$ and $y$ denote ranks, where by rank we mean order of magnitude or importance. That is, rank refers to the position of a variate in an arrangement.

If $x$ and $y$ denote the ranks of the same item with respect to two characteristics, and no ranks are omitted, and there are no duplications of ranks, then both $x$ and $y$ refer to the integers from 1 to $N$. Therefore, $\bar{x} = \bar{y}$, and $\sigma_x^2 = \frac{1}{12}(N^2 - 1) = \sigma_y^2$. See Theorem VI, Chapter V. Moreover,

$$\sigma_z^2 = \frac{1}{N}\sum z^2 - \bar{z}^2$$

$$= \frac{1}{N}\sum(x - y)^2 - (\bar{x} - \bar{y})^2$$

$$= \frac{1}{N}\sum(x - y)^2, \text{ since } \bar{x} - \bar{y} = 0.$$

Let $R$ denote the correlation coefficient when $x$ and $y$ refer to ranks

rather than variates. Then (49) becomes

$$R = \frac{2\left(\frac{N^2-1}{12}\right) - \frac{1}{N}\sum(x-y)^2}{2\left(\frac{N^2-1}{12}\right)}$$

which simplifies into

$$R = 1 - \frac{6\sum(x-y)^2}{N(N^2-1)}. \tag{50}$$

This is known as *Spearman's formula* for rank correlation.

If two or more variates are tied it is customary to divide the corresponding rank numbers among the variates concerned, using fractions if necessary.

*Example.* Suppose we have the following scores made in two tests, arranged in the order of their rank. Find the correlation between ranks.

| *Individual* | *1st Subject* | | *2nd Subject* | | $x-y$ | $(x-y)^2$ |
|---|---|---|---|---|---|---|
| | *Score* | *Rank* $= x$ | *Score* | *Rank* $= y$ | | |
| *A* | 92 | 1 | 85 | 2 | −1 | 1 |
| *B* | 86 | 2 | 76 | 4 | −2 | 4 |
| *C* | 84 | 3 | 93 | 1 | 2 | 4 |
| *D* | 78 | 4 | 68 | 6 | −2 | 4 |
| *E* | 71 | 5 | 67 | 7 | −2 | 4 |
| *F* | 69 | 6 | 83 | 3 | 3 | 9 |
| *G* | 66 | 7 | 54 | 9 | −2 | 4 |
| *H* | 58 | 8 | 70 | 5 | 3 | 9 |
| *I* | 53 | 9 | 43 | 10 | −1 | 1 |
| *J* | 45 | 10 | 59 | 8 | 2 | 4 |
| $N = 10$ | | | | | Total | 44 |

We find $R \quad 1 - \frac{6(44)}{10(99)} = .733.$

## Exercises

1. Suppose $z = x + y$. How would this change formulas (48) and (49)?
2. Twelve salesmen are ranked in order of merit for efficiency by their manager. They are also ranked in accordance with their length of service. What indication is there of a relation between length of service and efficiency? (Garrett.)

| *Salesmen* | *Years of Service* | *Order of Merit (Service)* | *Order of Merit (Effic.)* |
|---|---|---|---|
| *A* | 5 | 7.5 | 6 |
| *B* | 2 | 11.5 | 12 |
| *C* | 10 | 2 | 1 |
| *D* | 8 | 4 | 9 |
| *E* | 6 | 6 | 8 |
| *F* | 4 | 9 | 5 |
| *G* | 12 | 1 | 2 |
| *H* | 2 | 11.5 | 10 |
| *I* | 7 | 5 | 3 |
| *J* | 5 | 7.5 | 7 |
| *K* | 9 | 3 | 4 |
| *L* | 3 | 10 | 11 |

The fractions in the third column denote ties in rank. Thus, *A* and *J* each served 5 years and each is ranked 7.5. The next individual is ranked 9. *Ans.* $R = .80$.

3. Find $R$ for the following data:

| | *Rank* | *Score* | *Rank* | *Score* |
|---|---|---|---|---|
| *A* | 1 | 92 | 2 | 88 |
| *B* | 2 | 89 | 4 | 85 |
| *C* | 3 | 87 | 1 | 93 |
| *D* | 4 | 86 | 6 | 79 |
| *E* | 5 | 83 | 7 | 70 |
| *F* | 6 | 77 | 3 | 87 |
| *G* | 7 | 71 | 9 | 52 |
| *H* | 8 | 62 | 5 | 84 |
| *I* | 9 | 53 | 10 | 41 |
| *J* | 10 | 40 | 8 | 64 |

*Ans.* $R = .733$.

**25. Interpretation. Common Elements.** Although statistical theory gives a description of the indicated relationship between two related variables, the interpretation of the results " abound in pitfalls easily overlooked by the unwary, while they are cantering gaily along upon their arithmetic."

The methodological side has been developed until we can find correlation coefficients by simply turning a crank, but the explanation of the meaning of the result after we find it, needs a brain. . . . No amount of mathematical training and ability can take the place of the judgment and common sense that comes from a knowledge of the field in which the problem lies.[1]

In the interpretation of $r$ one should avoid imputing any causal relationship between the variables. In this connection the following pungent remarks of Professor E. B. Wilson[2] may be appropriately quoted:

Correlation is a mutual affair between two numerical variables; the correlation coefficient $r$ is symmetrical with respect to them. Strictly, $y$ is not correlated with $x$ or $x$ with $y$, but $x$ and $y$ are correlated. Theory is very important in indicating what facts should be looked for as significant; facts are significant or important largely as they indicate theory, but neither compels the other, as the histories of theorizing and of fact finding amply demonstrate . . . Further, the value of the correlation coefficient depends on the group for which it is determined or on the universe of which that group is a fair sample. The correlation coefficient $r$ of height and weight for a group containing humans from infancy to adult life would be different from, and in fact greater than, the coefficient for college students or for the members of a football squad; there is no such thing as the correlation coefficient per se.

If the student has mastered the underlying mathematical theory he should be able to understand and profit by the interpretations given by the writers in his particular field of interest. As a final aid in forming a conception of its meaning, we state a theorem which gives to $r$ a meaning in pure chance. If $x$ and $y$ are affected by $s$ equally likely causes of which $t$ are common to both, then $r = t/s$.

**Theorem VI.** *An urn containing white and black balls is so maintained that in drawing a ball the probability of getting a white ball is a constant $p$ and that of getting a black ball is $q$ $(= 1 - p)$. The first drawing of a pair of drawings is to consist of $s$ balls taken one at a time from the urn. The second drawing is to consist of $s$ balls of which $t$ are taken at random from the $s$ first drawn, and $s - t$ are drawn one at a time from the urn. Then the correlation coefficient between the numbers of white balls in the two drawings is $t/s$.*

As an illustration of the theorem we will take $s = 5$, $t = 3$, $p = \frac{1}{2}$. Let $x$ be the number of white balls in the first drawing and $y$ the

[1] Crathorne in Journal of the American Statistical Association, vol. 26, Supplement, March, 1931, p. 27.

[2] *Correlation and Association*, Journal of the American Statistical Association, vol. 26 (1931), pp. 250–256.

number of white balls in the second drawing. Then Table 37, constructed by the theory of probability,[1] exhibits the *a priori* frequencies when we use as small numbers as possible for frequencies subject to the condition that each frequency is to be an integer.

TABLE 37 — A PRIORI FREQUENCIES

| $y$ \ $x$ | 0 | 1 | 2 | 3 | 4 | 5 | $f(y)$ |
|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | 0 | 9 | 6 | 1 | 16 |
| 4 | 0 | 0 | 81 | 108 | 45 | 6 | 240 |
| 3 | 0 | 243 | 648 | 432 | 108 | 9 | 1440 |
| 2 | 243 | 1620 | 1728 | 648 | 81 | 0 | 4320 |
| 1 | 1458 | 3159 | 1620 | 243 | 0 | 0 | 6480 |
| 0 | 2187 | 1458 | 243 | 0 | 0 | 0 | 3888 |
| $f(x)$ | 3888 | 6480 | 4320 | 1440 | 240 | 16 | 16,384 |

According to the theorem the correlation coefficient should be $\frac{3}{5}$. It is left as an exercise for the student to show, by computing $r$ from the table, that this is actually the case.

## Review Questions and Problems

1. Define the following terms: statistics, variate, discrete, class interval, class mark, $x$-array of $y$'s, range, regression line, sample, universe, coefficient of variation, variance.
2. Name and define five averages. Discuss their advantages and limitations.
3. What does a ratio chart show that a chart with a uniform scale does not? If you wished to plot data so as to secure the effect of a ratio chart, but had no ratio paper available, how would you accomplish the desired result?
4. Prove the following:
   (*a*) The algebraic sum of the deviations of the variates from their mean is zero.
   (*b*) The second moment about an arbitrary point equals the second moment about the mean increased by the square of the distance between the arbitrary point and the mean.

[1] Explained in Part II.

5. (*a*) Define and explain how to compute the following:

$$Q_1,\ Q_2,\ Q,\ \text{MD},\ s,\ \sigma.$$

(*b*) In the case of a normal distribution give the value of each of the first four constants in (*a*) in terms of $\bar{x}$ or $\sigma$.

6. (*a*) Give the equation of the normal curve in both arbitrary coördinates and standard units. State the relation between abscissas and between ordinates in the two systems.

(*b*) State the properties of the normal curve.

7. Show how to fit a straight line $y = mx + k$ by the method of moments by deriving the expressions for $m$ and $k$.

8. Show how to fit an exponential function by the method explained in the text.

9. Show how to fit a parabola by the method of moments.

10. (*a*) Give two of the formulas for $r$. Discuss the use or uses of correlation in any problem that occurs to you.

(*b*) Show that the slope of the line in problem 7 may be written $r\sigma_y/\sigma_x$.

11. Prove that $|r| \leq 1$.

(*b*) Define the correlation ratio. Discuss its use.

12. Discuss rank correlation.

13. Derive the following relations:

$$\bar{x} = c\bar{u} + x_0$$
$$\mu_2 = \nu_2 - \nu_1^2$$
$$\mu_{2:x} = c^2\mu_{2:u}$$
$$\sigma_x = c\sigma_u.$$

14. The following is a reduced distribution of the breakfast checks at a cafeteria. Using the indirect method find $\bar{x}$ and $\sigma_x$.

| $x$ | $f$ |
|---|---|
| 8–12 | 4 |
| 13–17 | 8 |
| 18–22 | 24 |
| 23–27 | 21 |
| 28–32 | 15 |
| 33–37 | 14 |
| 38–42 | 7 |
| 43–47 | 4 |
| 48–52 | 2 |
| 53–57 | 1 |

*Ans.* $\bar{x} = 27.2¢$, $\sigma = 9.4¢$.

15. Derive the relations which give the third and fourth moments about the mean in terms of moments about an arbitrary origin. Define $\alpha_3$ and $\alpha_4$. What information do they give?

16. Compute the value of $\alpha_3$ and of $\alpha_4$ for the distribution in Exercise 14.

**17.** The following is a distribution of the heights of students where $x$ denotes heights in inches and $f$ is the number of students of the corresponding heights. Find $\bar{x}$, $\sigma_x$, $\alpha_3$, and $\alpha_4$.

| $x$ | $f$ |
|---|---|
| 60.5 | 1 |
| 62.0 | 3 |
| 63.5 | 14 |
| 65.0 | 32 |
| 66.5 | 61 |
| 68.0 | 80 |
| 69.5 | 71 |
| 71.0 | 35 |
| 72.5 | 24 |
| 74.0 | 2 |
| 75.5 | 1 |

**18.** For $N$ values of a variable $v$ it is known that $\sum v = 0$ and $\sum v^2 = N$. What are the origin and unit of $v$?

**19.** Find in two ways the value of $P$ for which the function

$$y = \frac{1}{N}\sum f(x - P)^2$$

has the smallest value.

**20.** (*Walker*) An algebra test was given to 400 high school children, of whom 150 were boys and 250 were girls. The results were as follows:

$$n_1 = 150 \qquad n_2 = 250$$
$$\bar{x}_1 = 72.5 \qquad \bar{x}_2 = 73.6$$
$$\sigma_1 = 7.0 \qquad \sigma_2 = 6.4$$

Find the mean and standard deviation of the combined groups.

**21.** For a normal distribution of 1500 students' grades, $\bar{x} = 75$, $\sigma_x = 10$. What values of $x$ will include the middle 500 grades? How many grades were below 60; above 90?

**22.** Suppose a distribution of 1000 breakfast checks from the cafeteria mentioned in problem 14 showed the following results: $\bar{x} = 27¢$, $\sigma_x = 9¢$, $\alpha_3 = 0$, $\alpha_4 = 3$. On the basis of these results what is the expected frequency in the 23-27¢ class interval?

**23.** Given the following data as to the heights ($y$) and weights ($x$) of college men:

$$\sum y = 6{,}800, \qquad \sum y^2 = 463{,}025, \qquad \sum xy = 1{,}022{,}250$$
$$\sum x = 15{,}000, \qquad \sum x^2 = 2{,}272{,}500, \qquad N = 100.$$

Find $\bar{x}$, $\bar{y}$, $\alpha_x$, $\sigma_y$, $r$.

**24.** Derive the expression for the standard error of estimate,

$$S_y = \sigma_y(1 - r^2)^{1/2}.$$

**25.** Discuss the use of $S_y$ in predictions.

**26.** Compute the median, quartiles, and quartile deviation for the following distribution where $x$ = bushels per acre and $f$ = corresponding frequency.

| $x$ | $f$ |
|---|---|
| 1 | 3 |
| 3 | 26 |
| 5 | 78 |
| 7 | 107 |
| 9 | 113 |
| 11 | 65 |
| 13 | 40 |
| 15 | 22 |
| 17 | 45 |
| 19 | 41 |
| 21 | 21 |
| 23 | 23 |

**27.** (*a*) Find $r$ for the following table using $(u, v)$ coördinates.

| $y$ \ $x$ | 17 | 19 | 21 | 23 | $f(y)$ |
|---|---|---|---|---|---|
| 18 | | 3 | 2 | 1 | 6 |
| 15 | 2 | 4 | 3 | 1 | 10 |
| 12 | 2 | 1 | 1 | | 4 |
| $f(x)$ | 4 | 8 | 6 | 2 | 20 |

(*b*) For the above data, find $\bar{x}$, $\bar{y}$, $\sigma_x$, $\sigma_y$, and the equations of the regression lines.

**28.** For Table 38, (*a*) find the correlation coefficient, (*b*) find the equations of the lines of regression, (*c*) locate the coördinate axes through the arithmetic mean of the table and plot the lines obtained in (*b*).

**29.** Fit an exponential function of the type $y = Ae^{Bx}$ to the following data:

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 2 | 10 | 100 |

First find the equation in the forms

(*a*) $Y = at + b$

(*b*) $Y = mx + k$

and then determine $A$ and $B$.

TABLE 38 — CORRELATION TABLE FOR MONTHLY RAINFALL AT IOWA CITY AND DES MOINES, 1890–1925

IOWA CITY

| DES MOINES $y$ \ $x$ | 0.245 | 0.745 | 1.245 | 1.745 | 2.245 | 2.745 | 3.245 | 3.745 | 4.245 | 4.745 | 5.245 | 5.745 | 6.245 | 6.745 | 7.245 | 7.745 | 8.245 | 8.745 | 9.245 | 9.745 | 10.245 | 10.745 | $f(y)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10.245 | | | | | | | | | | | | | | | | | | | 1 | | | | 1 |
| 9.745 | | | | | | | | | | | | | | | | 1 | | | | | | | 1 |
| 9.245 | | | | | | | | | | | | | | | | | 1 | 1 | | | | | 2 |
| 8.745 | | | | | | | | | | | | | 1 | | | | | | 1 | | | | 2 |
| 8.245 | | | | | | | | | | | | | 2 | | 1 | 1 | | | | | | | 4 |
| 7.745 | | | | | | | | | | | 1 | | | | | | | | | | | 1 | 2 |
| 7.245 | | | | | | | | | | 2 | 1 | | 1 | | 2 | 1 | | | | | | | 7 |
| 6.745 | | | 2 | | 1 | | 2 | | 1 | | | 1 | | | 1 | | | 1 | 1 | | | | 10 |
| 6.245 | | | | | | | | | | | | 1 | | | | | | | | | | | 1 |
| 5.745 | | | | | | 4 | 1 | | 1 | | | | 1 | 2 | | | 1 | | | | | | 10 |
| 5.245 | | | 1 | | 1 | 2 | | | 2 | | 2 | 1 | | 1 | 1 | | | | 1 | | | | 12 |
| 4.745 | | | | | 2 | 1 | 2 | 2 | 1 | 1 | 1 | | 1 | | | 2 | | | 1 | | | | 14 |
| 4.245 | | | 1 | | 1 | 1 | 2 | | 1 | 4 | 1 | 2 | 1 | | | | | | | | | | 14 |
| 3.745 | | | | 4 | 2 | 1 | 2 | 6 | 5 | 3 | 2 | 2 | 2 | | 1 | | | | | | | | 30 |
| 3.245 | | 2 | 1 | 4 | 6 | 6 | 3 | 7 | 2 | | 1 | | | | | | 1 | | | | 1 | | 34 |
| 2.745 | | | 3 | 4 | 1 | 8 | 4 | 4 | 2 | 1 | 2 | | | 1 | | | | | | | | | 30 |
| 2.245 | | | 1 | 5 | 10 | 7 | 6 | 4 | 2 | 2 | | | | | | | | | | | | | 37 |
| 1.745 | 1 | 4 | 7 | 12 | 13 | 8 | 5 | 1 | 1 | 1 | 2 | | 1 | | | | | | | | | | 56 |
| 1.245 | 3 | 8 | 18 | 17 | 6 | 8 | 4 | 2 | | 1 | | | | | | | | | | | | | 67 |
| 0.745 | 6 | 16 | 21 | 12 | 6 | 1 | 1 | 1 | | | 1 | | | 1 | | | | | | | | | 66 |
| 0.245 | 13 | 12 | 3 | 4 | | | | | | | | | | | | | | | | | | | 32 |
| $f(x)$ | 23 | 42 | 58 | 62 | 49 | 47 | 32 | 27 | 18 | 15 | 14 | 7 | 10 | 5 | 6 | 5 | 3 | 2 | 5 | 0 | 1 | 1 | 432 |

**30.** How does the scatter diagram assist one in deciding whether the regression is linear or non-linear? Give the formulas for the correlation coefficient and for the correlation ratio of $y$ on $x$, explaining the meaning of the letters used. How would you use these indices of correlation to decide whether the regression of $y$ on $x$ is linear or non-linear?

**31.** (*a*) In a normal distribution in which $\bar{x} = 0$ and $\sigma_x = 4$, what proportion of the data lie where $x > 12$?

(*b*) If 100 of the data lie between $x = -6$ and $x = -8$, how many of the data are there in the whole distribution?

**32.** (*a*) When the variates are ungrouped what is perhaps the best formula for $\sigma_x$? *Ans.*

$$\sigma_x = \frac{[N\sum x^2 - (\sum x)^2]^{1/2}}{N}.$$

(*b*) What does this expression become in terms of $N$ when $x$ refers to the integers from 1 to $N$?

**33.** (*a*) Expand $(a + b + c + d)^2$.

(*b*) The expansion of $(x_1 + x_2 + \cdots + x_n)^2$ consists of the sum of the squares of the $x$'s plus the sum of their products taken two at a time. Express this expansion in summation notation.

**34.** (*a*) Show that the formula for MD may be written

$$\text{MD} = \frac{2}{N}[\bar{x}\sum_{x_i<\bar{x}} f_i - \sum_{x_i<\bar{x}} f_i x_i].$$

*Hint.* For $x_i < \bar{x}$, $\sum f_i|x_i - \bar{x}| = -\sum f_i(x_i - \bar{x}) = \sum f_i(\bar{x} - x_i) = \bar{x}\sum f_i - \sum f_i x_i$.

For $x_i > \bar{x}$, $\sum f_i|x_i - \bar{x}| = -\sum f_i(\bar{x} - x_i)$.

Since $\bar{x}$ is the centroid (§14, Chapter III), $-\sum f_i(\bar{x} - x_i)$ for $x_i > \bar{x}$ equals $\sum f_i(\bar{x} - x_i)$ for $x_i < \bar{x}$.

(*b*) Using this formula evaluate MD for one of the distributions in the text.

**35.** Given $N$ pairs of variates: $(x_{11}, x_{21})$; $(x_{12}, x_{22})$; $(x_{13}, x_{23})$; $\cdots$; $(x_{1n}, x_{2n})$. Show that:

(*a*) the mean $\bar{x}$ of all the variates is

$$\bar{x} = \frac{1}{2N}\sum_1^n (x_{1i} + x_{2i}),$$

(*b*) the variance $\sigma^2$ taken about the $\bar{x}$ in (*a*) is

$$\sigma^2 = \frac{1}{2N}[\sum_1^n (x_{1i} - \bar{x})^2 + \sum_1^n (x_{2i} - \bar{x})^2].$$

*Note.* The quantity

$$r' = \frac{1}{N\sigma^2}\sum_1^n (x_{1i} - \bar{x})(x_{2i} - \bar{x})$$

where $\bar{x}$ and $\sigma^2$ are defined as in (*a*) and (*b*) is called the *intra-class* correlation coefficient. For its use see *Statistical Methods for Research Workers*, Fisher (§38), Oliver and Boyd, London.

**36.** Let $S_r = \frac{1}{N}\sum_{x=1}^{N} x^r$. Prove that $S_1 = N(N+1)/2$, $S_2 = N(N+1)(2N+1)/6$, $S_3 = S_1^2$.

**37.** Sketch the graph of $y = Ae^{Bx}$, $-\infty \leq x \leq \infty$, when (*a*) both $A$ and $B$ are positive, (*b*) $A$ is positive and $B$ negative, (*c*) $A$ is negative and $B$ positive, (*d*) both $A$ and $B$ are negative.

**38.** A large number of rectangles are drawn all having the same perimeter but different bases ($x$) and altitudes ($y$). Which of the following is the correct answer? The coefficient of correlation between $x$ and $y$ is (*a*) negative and numerically large, (*b*) positive and numerically small, (*c*) positive and numerically large, (*d*) approximately zero.

**39.** For $N$ correlated values of $x$ and $y$ the regression equation of $y$ on $x$ is found to be $y = 1 + x$. If $\bar{x} = 0$, $r = 0.5$, and $\sigma_x = 1$, determine $\bar{y}$ and $S_y$.

**40.** Let $NS_y^2$ denote the sum of squares of deviations from the line of least squares (Case I).

(*a*) Show that $NS_y^2 = \sum y^2 - m\sum xy - k\sum y$.

*Hint.*
$$NS_y^2 = \sum(y - mx - k)^2$$
$$= \sum y(y - mx - k) - m\sum x(y - mx - k)$$
$$\therefore \quad - k\sum(y - mx - k).$$

The last two expressions vanish. Why?

(*b*) If $m$ and $k$ are replaced by their determinant values from (5), p. 143, show that

$$NS_y^2 = \frac{1}{D}\begin{vmatrix} \sum y^2 & \sum y & \sum xy \\ \sum y & N & \sum x \\ \sum xy & \sum x & \sum x^2 \end{vmatrix}, \qquad D = \begin{vmatrix} N & \sum x \\ \sum x & \sum x^2 \end{vmatrix}.$$

The third order determinant is $D$ bordered by $\sum y^2$, $\sum y$, $\sum xy$.

(*c*) If $x$ and $y$ are replaced by $x'$ and $y'$, denoting deviations from their respective means, find the values of the resulting determinants in (*b*).

(*d*) From the results in (*c*) show that $S_y^2 = \sigma_y^2(1 - r^2)$.

**41.** Discuss the properties of the normal correlation surface and their use in passing judgment on the reliability of predictions based upon the regression line of $y$ on $x$.

**42.** (*For calculus students*) In fitting points in a plane by a line so that the sum of squares of perpendicular deviations shall be a minimum, a second line may be found for which the sum of squares of perpendicular deviations is a maximum. If $\sum d^2$ is the sum of squares of deviations from the first line and $\sum D^2$ is the sum of squares of deviations from the second line, show that $\sum d^2/\sum D^2 = (1 + r)/(1 - r)$. [Reference: *Bulletin American Mathematical Society*, vol. 47 (1941), p. 710.]

# APPENDIX

## Tables

I. Ordinates and Areas of the Normal Curve.

II. Common Logarithms of Numbers to Five Decimal Places.

TABLE I. ORDINATES AND AREAS OF THE NORMAL CURVE, $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

| $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ | $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ | $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ |
|---|---|---|---|---|---|---|---|---|
| .00 | .39894 | .00000 | .45 | .36053 | .17364 | .90 | .26609 | .31594 |
| .01 | .39892 | .00399 | 46 | .35889 | .17724 | .91 | .26369 | .31859 |
| .02 | .39886 | .00798 | .47 | .35723 | .18082 | .92 | .26129 | .32121 |
| .03 | .39876 | .01197 | .48 | .35553 | .18439 | .93 | .25888 | .32381 |
| .04 | .39862 | .01595 | .49 | .35381 | .18793 | .94 | .25647 | .32639 |
| .05 | .39844 | .01994 | .50 | .35207 | .19146 | .95 | .25406 | .32894 |
| .06 | .39822 | .02392 | .51 | .35029 | .19497 | .96 | .25164 | .33147 |
| .07 | .39797 | .02790 | .52 | .34849 | .19847 | .97 | .24923 | .33398 |
| .08 | .39767 | .03188 | .53 | .34667 | .20194 | .98 | .24681 | .33646 |
| .09 | .39733 | .03586 | .54 | .34482 | .20540 | .99 | .24439 | .33891 |
| .10 | .39695 | .03983 | .55 | .34294 | .20884 | 1.00 | .24197 | .34134 |
| .11 | .39654 | .04380 | .56 | .34105 | .21226 | 1.01 | .23955 | .34375 |
| .12 | .39608 | .04776 | .57 | .33912 | .21566 | 1.02 | .23713 | .34614 |
| .13 | .39559 | .05172 | .58 | .33718 | .21904 | 1.03 | .23471 | .34850 |
| .14 | .39505 | .05567 | .59 | .33521 | .22240 | 1.04 | .23230 | .35083 |
| .15 | .39448 | .05962 | .60 | .33322 | .22575 | 1.05 | .22988 | .35314 |
| .16 | .39387 | .06356 | .61 | .33121 | 22907 | 1.06 | .22747 | .35543 |
| .17 | .39322 | .06749 | .62 | .32918 | .23237 | 1.07 | .22506 | .35769 |
| .18 | .39253 | .07142 | 63 | .32713 | .23565 | 1.08 | .22265 | .35993 |
| .19 | .39181 | .07535 | 64 | .32506 | .23891 | 1.09 | .22025 | .36214 |
| .20 | .39104 | .07926 | .65 | 32297 | .24215 | 1.10 | .21785 | .36433 |
| .21 | .39024 | .08317 | .66 | .32086 | .24537 | 1.11 | .21546 | .36650 |
| .22 | .38940 | .08706 | .67 | .31874 | .24857 | 1.12 | .21307 | .36864 |
| .23 | .38853 | .09095 | .68 | .31659 | .25175 | 1.13 | .21069 | .37076 |
| .24 | .38762 | .09483 | .69 | .31443 | .25490 | 1.14 | .20831 | .37286 |
| .25 | .38667 | .09871 | .70 | .31225 | .25804 | 1.15 | .20594 | .37493 |
| .26 | .38568 | .10257 | .71 | .31006 | .26115 | 1.16 | .20357 | .37698 |
| .27 | .38466 | .10642 | .72 | .30785 | .26424 | 1.17 | .20121 | .37900 |
| .28 | .38361 | .11026 | .73 | .30563 | .26730 | 1.18 | .19886 | .38100 |
| .29 | .38251 | .11409 | .74 | .30339 | .27035 | 1.19 | .19652 | .38298 |
| .30 | .38139 | .11791 | .75 | .30114 | .27337 | 1 20 | .19419 | .38493 |
| .31 | .38023 | .12172 | .76 | .29887 | .27637 | 1.21 | .19186 | .38686 |
| .32 | .37903 | .12552 | .77 | .29659 | .27935 | 1.22 | .18954 | .38877 |
| .33 | .37780 | .12930 | .78 | .29431 | .28230 | 1.23 | .18724 | .39065 |
| .34 | .37654 | .13307 | .79 | .29200 | .28524 | 1.24 | .18494 | .39251 |
| .35 | .37524 | .13683 | .80 | .28969 | .28814 | 1.25 | .18265 | .39435 |
| .36 | .37391 | .14058 | .81 | .28737 | .29103 | 1.26 | .18037 | .39617 |
| .37 | .37255 | .14431 | .82 | .28504 | .29389 | 1.27 | .17810 | .39796 |
| .38 | .37115 | .14803 | .83 | .28269 | .29673 | 1.28 | .17585 | .39973 |
| .39 | .36973 | .15173 | .84 | .28034 | .29955 | 1.29 | .17360 | .40147 |
| .40 | .36827 | .15542 | .85 | .27798 | .30234 | 1.30 | .17137 | .40320 |
| .41 | .36678 | .15910 | .86 | .27562 | .30511 | 1.31 | .16915 | .40490 |
| .42 | .36526 | .16276 | .87 | .27324 | .30785 | 1.32 | .16694 | .40658 |
| .43 | .36371 | .16640 | .88 | .27086 | .31057 | 1.33 | .16474 | .40824 |
| .44 | .36213 | .17003 | .89 | .26848 | .31327 | 1.34 | .16256 | .40988 |

TABLE I. ORDINATES AND AREAS OF THE NORMAL CURVE, $\phi(t) \quad \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

| $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ | $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ | $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ |
|---|---|---|---|---|---|---|---|---|
| 1.35 | .16038 | .41149 | 1.80 | .07895 | .46407 | 2.25 | .03174 | .48778 |
| 1.36 | .15822 | .41309 | 1.81 | .07754 | .46485 | 2.26 | .03103 | .48809 |
| 1.37 | .15608 | .41466 | 1.82 | .07614 | .46562 | 2.27 | .03034 | .48840 |
| 1.38 | .15395 | .41621 | 1.83 | .07477 | .46638 | 2.28 | .02965 | .48870 |
| 1.39 | .15183 | .41774 | 1.84 | .07341 | .46712 | 2.29 | .02898 | .48899 |
| 1.40 | .14973 | .41924 | 1.85 | .07206 | .46784 | 2.30 | .02833 | .48928 |
| 1.41 | .14764 | .42073 | 1.86 | .07074 | .46856 | 2.31 | .02768 | .48956 |
| 1.42 | .14556 | .42220 | 1.87 | .06943 | .46926 | 2.32 | .02705 | .48983 |
| 1.43 | .14350 | .42364 | 1.88 | .06814 | .46995 | 2.33 | .02643 | .49010 |
| 1.44 | .14146 | .42507 | 1.89 | .06687 | .47062 | 2.34 | 02582 | .49036 |
| 1.45 | .13943 | .42647 | 1.90 | .06562 | .47128 | 2.35 | .02522 | .49061 |
| 1.46 | .13742 | .42786 | 1.91 | .06439 | .47193 | 2.36 | .02463 | .49086 |
| 1.47 | .13542 | .42922 | 1.92 | 06316 | .47257 | 2.37 | .02406 | .49111 |
| 1.48 | .13344 | .43056 | 1.93 | .06195 | .47320 | 2.38 | .02349 | .49134 |
| 1.49 | .13147 | .43189 | 1.94 | .06077 | .47381 | 2.39 | .02294 | .49158 |
| 1.50 | .12952 | .43319 | 1.95 | .05959 | .47441 | 2 40 | .02239 | .49180 |
| 1.51 | .12758 | .43448 | 1.96 | .05844 | .47500 | 2.41 | .02186 | .49202 |
| 1.52 | .12566 | .43574 | 1.97 | .05730 | .47558 | 2.42 | .02134 | .49224 |
| 1.53 | .12376 | .43699 | 1.98 | .05618 | .47615 | 2.43 | 02083 | .49245 |
| 1 54 | .12188 | .43822 | 1.99 | .05508 | .47670 | 2.44 | .02033 | .49266 |
| 1.55 | .12001 | .43943 | 2 00 | .05399 | .47725 | 2 45 | .01984 | .49286 |
| 1.56 | .11816 | .44062 | 2 01 | .02592 | 47778 | 2 46 | .01936 | .49305 |
| 1.57 | .11632 | .44179 | 2 02 | .05186 | .47831 | 2 47 | .01889 | .49324 |
| 1.58 | .11450 | .44295 | 2.03 | .05082 | .47882 | 2.48 | .01842 | .49343 |
| 1.59 | .11270 | .44408 | 2.04 | .04980 | .47932 | 2.49 | 01797 | .49361 |
| 1.60 | .11092 | .44520 | 2.05 | .04879 | .47982 | 2.50 | .01753 | .49379 |
| 1.61 | .10915 | .44630 | 2.06 | .04780 | .48030 | 2.51 | .01709 | .49396 |
| 1.62 | .10741 | .44738 | 2.07 | .04682 | .48077 | 2.52 | .01667 | .49413 |
| 1.63 | .10567 | .44845 | 2.08 | .04586 | .48124 | 2.53 | .01625 | .49430 |
| 1.64 | .10396 | .44950 | 2.09 | .04491 | .48169 | 2.54 | .01585 | .49446 |
| 1.65 | .10226 | .45053 | 2.10 | .04398 | .48214 | 2.55 | .01545 | .49461 |
| 1.66 | .10059 | .45154 | 2.11 | .04307 | .48257 | 2.56 | .01506 | .49477 |
| 1.67 | .09893 | .45254 | 2.12 | .04217 | .48300 | 2.57 | .01468 | .49492 |
| 1.68 | .09728 | .45352 | 2.13 | .04128 | .48341 | 2.58 | .01431 | .49506 |
| 1.69 | .09566 | .45449 | 2.14 | .04041 | .48382 | 2.59 | .01394 | .49520 |
| 1.70 | .09405 | .45543 | 2.15 | .03955 | .48422 | 2.60 | .01358 | .49534 |
| 1.71 | .09246 | .45637 | 2.16 | .03871 | .48461 | 2.61 | .01323 | .49547 |
| 1.72 | .09089 | .45728 | 2.17 | .03788 | .48500 | 2.62 | .01289 | .49560 |
| 1.73 | .08933 | .45818 | 2.18 | .03706 | .48537 | 2.63 | .01256 | .49573 |
| 1.74 | .08780 | .45907 | 2.19 | .03626 | .48574 | 2.64 | .01223 | .49585 |
| 1.75 | .08628 | .45994 | 2.20 | .03547 | .48610 | 2.65 | .01191 | .49598 |
| 1.76 | .08478 | .46080 | 2.21 | .03470 | .48645 | 2.66 | .01160 | .49609 |
| 1.77 | .08329 | .46164 | 2.22 | .03394 | .48679 | 2.67 | .01130 | .49621 |
| 1.78 | .08183 | .46246 | 2.23 | .03319 | .48713 | 2.68 | .01100 | .49632 |
| 1.79 | .08038 | .46327 | 2.24 | .03246 | .48745 | 2.69 | .01071 | .49643 |

TABLE I. ORDINATES AND AREAS OF THE NORMAL CURVE, $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

| $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ | $t$ | $\phi(t)$ | $\int_0 \phi(t)dt$ | $t$ | $\phi(t)$ | $\int_0^t \phi(t)dt$ |
|---|---|---|---|---|---|---|---|---|
| 2.70 | .01042 | .49653 | 3.15 | .00279 | .49918 | 3.60 | .00061 | .49984 |
| 2.71 | .01014 | .49664 | 3.16 | .00271 | .49921 | 3.61 | .00059 | .49985 |
| 2.72 | .00987 | .49674 | 3.17 | .00262 | .49924 | 3.62 | .00057 | .49985 |
| 2.73 | .00961 | .49683 | 3.18 | .00254 | .49926 | 3.63 | .00055 | .49986 |
| 2.74 | .00935 | .49693 | 3.19 | .00246 | .49929 | 3.64 | .00053 | .49986 |
| 2.75 | .00909 | .49702 | 3.20 | .00238 | .49931 | 3.65 | .00051 | .49987 |
| 2.76 | .00885 | .49711 | 3.21 | .00231 | .49934 | 3 66 | .00049 | .49987 |
| 2.77 | .00861 | .49720 | 3.22 | .00224 | .49936 | 3.67 | .00047 | .49988 |
| 2.78 | .00837 | .49728 | 3.23 | .00216 | .49938 | 3.68 | .00046 | .49988 |
| 2.79 | .00814 | .49736 | 3.24 | .00210 | .49940 | 3.69 | .00044 | .49989 |
| 2.80 | .00792 | .49744 | 3.25 | .00203 | .49942 | 3.70 | .00042 | .49989 |
| 2.81 | .00770 | .49752 | 3.26 | .00196 | .49944 | 3 71 | .00041 | .49990 |
| 2.82 | .00748 | .49760 | 3.27 | .00190 | .49946 | 3.72 | .00039 | .49990 |
| 2.83 | .00727 | .49767 | 3.28 | .00184 | .49948 | 3.73 | .00038 | .49990 |
| 2.84 | .00707 | .49774 | 3.29 | .00178 | .49950 | 3.74 | .00037 | .49991 |
| 2.85 | .00687 | .49781 | 3.30 | .00172 | 49952 | 3.75 | .00035 | .49991 |
| 2.86 | .00668 | .49788 | 3 31 | .00167 | .49953 | 3.76 | .00034 | .49992 |
| 2.87 | .00649 | .49795 | 3.32 | .00161 | .49955 | 3.77 | 00033 | .49992 |
| 2.88 | .00631 | .49801 | 3.33 | .00156 | .49957 | 3 78 | 00031 | .49992 |
| 2.89 | .00613 | .49807 | 3.34 | .00151 | .49958 | 3.79 | .00030 | .49992 |
| 2.90 | .00595 | 49813 | 3.35 | 00146 | 49960 | 3 80 | .00029 | .49993 |
| 2.91 | 00578 | 49819 | 3.36 | 00141 | .49961 | 3.81 | .00028 | .49993 |
| 2.92 | .00562 | .49825 | 3.37 | 00136 | .49962 | 3 82 | .00027 | .49993 |
| 2.93 | .00545 | .49831 | 3.38 | .00132 | .49964 | 3.83 | .00026 | 49994 |
| 2.94 | .00530 | .49836 | 3.39 | .00127 | 49965 | 3.84 | 00025 | .49994 |
| 2.95 | .00514 | .49841 | 3.40 | .00123 | .49966 | 3.85 | .00024 | .49994 |
| 2.96 | .00499 | .49846 | 3 41 | .00119 | 49968 | 3.86 | .00023 | .49994 |
| 2.97 | .00485 | .49851 | 3 42 | .00115 | .49969 | 3.87 | .00022 | .49995 |
| 2.98 | .00471 | .49856 | 3.43 | .00111 | .49970 | 3.88 | .00021 | .49995 |
| 2.99 | .00457 | .49861 | 3.44 | .00107 | .49971 | 3.89 | .00021 | .49995 |
| 3.00 | .00443 | .49865 | 3.45 | .00104 | .49972 | 3.90 | .00020 | .49995 |
| 3.01 | .00430 | .49869 | 3.46 | .00100 | .49973 | 3.91 | .00019 | .49995 |
| 3.02 | .00417 | .49874 | 3.47 | .00097 | .49974 | 3.92 | .00018 | .49996 |
| 3.03 | .00405 | .49878 | 3.48 | .00094 | .49975 | 3.93 | .00018 | .49996 |
| 3.04 | .00393 | .49882 | 3.49 | .00090 | 49976 | 3.94 | .00017 | .49996 |
| 3.05 | .00381 | .49886 | 3.50 | .00087 | .49977 | 3.95 | .00016 | .49996 |
| 3.06 | .00370 | .49889 | 3.51 | .00084 | .49978 | 3.96 | .00016 | .49996 |
| 3.07 | .00358 | .49893 | 3.52 | .00081 | .49978 | 3.97 | .00015 | .49996 |
| 3.08 | .00348 | .49897 | 3.53 | .00079 | .49979 | 3.98 | .00014 | .49997 |
| 3.09 | .00337 | .49900 | 3.54 | .00076 | .49980 | 3.99 | .00014 | .49997 |
| 3.10 | .00327 | .49903 | 3.55 | .00073 | .49981 | | | |
| 3.11 | .00317 | .49906 | 3.56 | .00071 | .49981 | | | |
| 3.12 | .00307 | .49910 | 3 57 | .00068 | .49982 | | | |
| 3.13 | .00298 | .49913 | 3.58 | .00066 | .49983 | | | |
| 3.14 | .00288 | .49916 | 3.59 | .00063 | .49983 | | | |

## Table II. Common Logarithms of Numbers to Five Decimal Places

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 00 000 | 043 | 087 | 130 | 173 | 217 | 260 | 303 | 346 | 389 | |
| 01 | 432 | 475 | 518 | 561 | 604 | 647 | 689 | 732 | 775 | 817 | 44 43 42 |
| 02 | 00 860 | 903 | 945 | 988 | *030 | *072 | *115 | *157 | *199 | *242 | 1 4.4 4.3 4.2 |
| 03 | 01 284 | 326 | 368 | 410 | 452 | 494 | 536 | 578 | 620 | 662 | 2 8.8 8.6 8.4 |
| 04 | 01 703 | 745 | 787 | 828 | 870 | 912 | 953 | 995 | *036 | *078 | 3 13.2 12.9 12.6 |
| 05 | 02 119 | 160 | 202 | 243 | 284 | 325 | 366 | 407 | 449 | 490 | 4 17.6 17.2 16.8 |
| 06 | 531 | 572 | 612 | 653 | 694 | 735 | 776 | 816 | 857 | 898 | 5 22.0 21.5 21.0 |
| 07 | 02 938 | 979 | *019 | *060 | *100 | *141 | *181 | *222 | *262 | *302 | 6 26.4 25.8 25.2 |
| 08 | 03 342 | 383 | 423 | 463 | 503 | 543 | 583 | 623 | 663 | 703 | 7 30.8 30.1 29.4 |
| 09 | 03 743 | 782 | 822 | 862 | 902 | 941 | 981 | *021 | *060 | *100 | 8 35.2 34.4 33.6 |
| | | | | | | | | | | | 9 39.6 38.7 37.8 |
| 110 | 04 139 | 179 | 218 | 258 | 297 | 336 | 376 | 415 | 454 | 493 | |
| 11 | 532 | 571 | 610 | 650 | 689 | 727 | 766 | 805 | 844 | 883 | |
| 12 | 04 922 | 961 | 999 | *038 | *077 | *115 | *154 | *192 | *231 | *269 | 41 40 39 |
| 13 | 05 308 | 346 | 385 | 423 | 461 | 500 | 538 | 576 | 614 | 652 | 1 4.1 4 3.9 |
| 14 | 05 690 | 729 | 767 | 805 | 843 | 881 | 918 | 956 | 994 | *032 | 2 8.2 8 7.8 |
| 15 | 06 070 | 108 | 145 | 183 | 221 | 258 | 296 | 333 | 371 | 408 | 3 12.3 12 11.7 |
| 16 | 446 | 483 | 521 | 558 | 595 | 633 | 670 | 707 | 744 | 781 | 4 16.4 16 15.6 |
| 17 | 06 819 | 856 | 893 | 930 | 967 | *004 | *041 | *078 | *115 | *151 | 5 20.5 20 19.5 |
| 18 | 07 188 | 225 | 262 | 298 | 335 | 372 | 408 | 445 | 482 | 518 | 6 24.6 24 23.4 |
| 19 | 555 | 591 | 628 | 664 | 700 | 737 | 773 | 809 | 846 | 882 | 7 28.7 28 27.3 |
| | | | | | | | | | | | 8 32.8 32 31.2 |
| 120 | 07 918 | 954 | 990 | *027 | *063 | *099 | *135 | *171 | *207 | *243 | 9 36.9 36 35.1 |
| 21 | 08 279 | 314 | 350 | 386 | 422 | 458 | 493 | 529 | 565 | 600 | |
| 22 | 636 | 672 | 707 | 743 | 778 | 814 | 849 | 884 | 920 | 955 | 38 37 36 |
| 23 | 08 991 | *026 | *061 | *096 | *132 | *167 | *202 | *237 | *272 | *307 | 1 3.8 3.7 3.6 |
| 24 | 09 342 | 377 | 412 | 447 | 482 | 517 | 552 | 587 | 621 | 656 | 2 7.6 7.4 7.2 |
| 25 | 09 691 | 726 | 760 | 795 | 830 | 864 | 899 | 934 | 968 | *003 | 3 11.4 11.1 10.8 |
| 26 | 10 037 | 072 | 106 | 140 | 175 | 209 | 243 | 278 | 312 | 346 | 4 15.2 14.8 14.4 |
| 27 | 380 | 415 | 449 | 483 | 517 | 551 | 585 | 619 | 653 | 687 | 5 19.0 18.5 18.0 |
| 28 | 10 721 | 755 | 789 | 823 | 857 | 890 | 924 | 958 | 992 | *025 | 6 22.8 22.2 21.6 |
| 29 | 11 059 | 093 | 126 | 160 | 193 | 227 | 261 | 294 | 327 | 361 | 7 26.6 25.9 25.2 |
| | | | | | | | | | | | 8 30.4 29.6 28.8 |
| 130 | 394 | 428 | 461 | 494 | 528 | 561 | 594 | 628 | 661 | 694 | 9 34.2 33.3 32.4 |
| 31 | 11 727 | 760 | 793 | 826 | 860 | 893 | 926 | 959 | 992 | *024 | |
| 32 | 12 057 | 090 | 123 | 156 | 189 | 222 | 254 | 287 | 320 | 352 | 35 34 33 |
| 33 | 385 | 418 | 450 | 483 | 516 | 548 | 581 | 613 | 646 | 678 | 1 3.5 3.4 3.3 |
| 34 | 12 710 | 743 | 775 | 808 | 840 | 872 | 905 | 937 | 969 | *001 | 2 7.0 6.8 6.6 |
| 35 | 13 033 | 066 | 098 | 130 | 162 | 194 | 226 | 258 | 290 | 322 | 3 10.5 10.2 9.9 |
| 36 | 354 | 386 | 418 | 450 | 481 | 513 | 545 | 577 | 609 | 640 | 4 14.0 13.6 13.2 |
| 37 | 672 | 704 | 735 | 767 | 799 | 830 | 862 | 893 | 925 | 956 | 5 17.5 17.0 16.5 |
| 38 | 13 988 | *019 | *051 | *082 | *114 | *145 | *176 | *208 | *239 | *270 | 6 21.0 20.4 19.8 |
| 39 | 14 301 | 333 | 364 | 395 | 426 | 457 | 489 | 520 | 551 | 582 | 7 24.5 23.8 23.1 |
| | | | | | | | | | | | 8 28.0 27.2 26.4 |
| 140 | 613 | 644 | 675 | 706 | 737 | 768 | 799 | 829 | 860 | 891 | 9 31.5 30.6 29.7 |
| 41 | 14 922 | 953 | 983 | *014 | *045 | *076 | *106 | *137 | *168 | *198 | |
| 42 | 15 229 | 259 | 290 | 320 | 351 | 381 | 412 | 442 | 473 | 503 | 33 32 31 |
| 43 | 534 | 564 | 594 | 625 | 655 | 685 | 715 | 746 | 776 | 806 | 1 3.3 3.2 3.1 |
| 44 | 15 836 | 866 | 897 | 927 | 957 | 987 | *017 | *047 | *077 | *107 | 2 6.6 6.4 6.2 |
| 45 | 16 137 | 167 | 197 | 227 | 256 | 286 | 316 | 346 | 376 | 406 | 3 9.9 9.6 9.3 |
| 46 | 435 | 465 | 495 | 524 | 554 | 584 | 613 | 643 | 673 | 702 | 4 13.2 12.8 12.4 |
| 47 | 16 732 | 761 | 791 | 820 | 850 | 879 | 909 | 938 | 967 | 997 | 5 16.5 16.0 15.5 |
| 48 | 17 026 | 056 | 085 | 114 | 143 | 173 | 202 | 231 | 260 | 289 | 6 19.8 19.2 18.6 |
| 49 | 319 | 348 | 377 | 406 | 435 | 464 | 493 | 522 | 551 | 580 | 7 23.1 22.4 21.7 |
| | | | | | | | | | | | 8 26.4 25.6 24.8 |
| 150 | 17 609 | 638 | 667 | 696 | 725 | 754 | 782 | 811 | 840 | 869 | 9 29.7 28.8 27.9 |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

## Table II. Common Logarithms of Numbers to Five Decimal Places

| Prop. Parts | N | | | | | 5 | 6 | | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 150 | | 638 | 667 | 696 | 725 | 754 | 782 | 811 | 840 | 869 |
| | | 17 898 | 926 | 955 | 98. | '01. | '041 | *070 | *099 | '127 | '156 |
| 29 28 | | 18 184 | 21. | 241 | 270 | 298 | 327 | 35. | 384 | 412 | 441 |
| 2.9 2.8 | | 469 | 498 | 526 | 554 | 583 | 611 | 639 | 66. | 696 | 724 |
| 5.8 5.6 | | | | | | | | | | | |
| 8.7 8.4 | | 18 752 | 780 | 808 | 83. | 865 | 893 | 921 | 949 | 97. | *005 |
| 11.6 11.2 | | 19 033 | 061 | 089 | 117 | 14. | 173 | 201 | 229 | 25. | 285 |
| 14.5 14.0 | | 312 | 340 | 368 | 396 | 424 | 451 | 479 | 507 | 535 | 562 |
| 17.4 16.8 | | | | | | | | | | | |
| 20.3 19.6 | | 590 | 618 | 645 | 673 | 700 | 728 | 756 | 783 | 811 | 838 |
| 23.2 22.4 | | 19 866 | 893 | 921 | 948 | 976 | *003 | *030 | '058 | | '112 |
| 26.1 25.2 | | 20 140 | 167 | 194 | 222 | 249 | 276 | 303 | 330 | | 385 |
| | 160 | 412 | 439 | 466 | 49. | 520 | 548 | 575 | 602 | 629 | 656 |
| | | 683 | 710 | 737 | 763 | 790 | 817 | 844 | 871 | 898 | 925 |
| 27 26 | | 20 952 | 978 | *005 | *032 | *059 | '085 | '112 | '139 | '165 | '192 |
| 2.7 2.6 | | 21 219 | 245 | 272 | 299 | 325 | 352 | 378 | 405 | 431 | 458 |
| 5.4 5.2 | | | | | | | | | | | |
| 8.1 7.8 | | 484 | 511 | 537 | 564 | 590 | 617 | 643 | 669 | 696 | 722 |
| 10.8 10.4 | | 21 748 | 775 | 801 | 827 | 854 | 880 | 906 | 932 | 958 | 985 |
| 13.5 13.0 | | 22 011 | 037 | 063 | 089 | 115 | 141 | 167 | 194 | 220 | 246 |
| 16.2 15.6 | | | | | | | | | | | |
| 18.9 18.2 | | 272 | 298 | 324 | 350 | 376 | 401 | 427 | 453 | 479 | 505 |
| 21.6 20.8 | | 531 | 557 | 583 | 608 | 634 | 660 | 686 | 712 | 737 | 763 |
| 24.3 23.4 | | 22 789 | 814 | 840 | 866 | 891 | 917 | 943 | 968 | 994 | *019 |
| | 170 | 23 045 | 070 | 096 | 121 | 147 | 172 | 198 | 223 | 249 | 274 |
| | | 300 | 325 | 350 | 376 | 401 | 426 | 452 | 477 | 502 | 528 |
| 25 | | 553 | 578 | 603 | 629 | 654 | 679 | 704 | 729 | 754 | 779 |
| 1 2.5 | | 23 805 | 830 | 855 | 880 | 905 | 930 | 955 | 980 | '005 | '030 |
| 2 5.0 | | | | | | | | | | | |
| 3 7.5 | | 24 055 | 080 | 105 | 130 | 155 | 180 | 204 | 229 | 254 | 279 |
| 4 10.0 | | 304 | 329 | 353 | 378 | 403 | 428 | 452 | 477 | 502 | 527 |
| 5 12.5 | | 551 | 576 | 601 | 625 | 650 | 674 | 699 | 724 | 748 | 773 |
| 6 15.0 | | | | | | | | | | | |
| 7 17.5 | | 24 797 | 822 | 846 | 871 | 895 | 920 | 944 | 969 | 993 | '018 |
| 8 20.0 | | 25 042 | 066 | 091 | 115 | 139 | 164 | 188 | 212 | 237 | |
| 9 22.5 | | 285 | 310 | 334 | 358 | 382 | 406 | 431 | 455 | 479 | |
| | 180 | 527 | 551 | 575 | 600 | 624 | 648 | 672 | 696 | 720 | |
| | | 25 768 | 792 | 816 | 840 | 864 | 888 | 912 | 935 | 959 | 983 |
| 24 23 | | 26 007 | 031 | 055 | 079 | 102 | 126 | 150 | 174 | 198 | 221 |
| 2.4 2.3 | | 245 | 269 | 293 | 316 | 340 | 364 | 387 | 411 | 435 | 458 |
| 4.8 4.6 | | | | | | | | | | | |
| 7.2 6.9 | | 482 | 505 | 529 | 553 | 576 | 600 | 623 | 647 | 670 | 694 |
| 9.6 9.2 | | 717 | 741 | 764 | 788 | 811 | 834 | 858 | 881 | 905 | 928 |
| 12.0 11.5 | | 26 951 | 975 | 998 | *021 | '045 | '068 | *091 | 114 | 138 | '161 |
| 14.4 13.8 | | | | | | | | | | | |
| 16.8 16.1 | | 27 184 | 207 | 231 | 254 | 277 | 300 | 323 | 346 | 370 | 393 |
| 19.2 18.4 | | 416 | 439 | 462 | 485 | 508 | 531 | 554 | 577 | 600 | 623 |
| 21.6 20.7 | | 646 | 669 | 692 | 715 | 738 | 761 | 784 | 807 | 830 | 852 |
| | 190 | 27 875 | 898 | 921 | 944 | 967 | 989 | *012 | '035 | '058 | '081 |
| | 91 | 28 103 | 126 | 149 | 171 | 194 | 217 | 240 | 262 | 285 | 307 |
| 22 21 | 92 | 330 | 353 | 375 | 398 | 421 | 443 | 466 | 488 | 511 | 533 |
| 2.2 2.1 | 93 | 556 | 578 | 601 | 623 | 646 | 668 | 691 | 713 | 735 | 758 |
| 4.4 4.2 | | | | | | | | | | | |
| 6.6 6.3 | 94 | 28 780 | 803 | 825 | 847 | 870 | 892 | 914 | 937 | 959 | 981 |
| 8.8 8.4 | 95 | 29 003 | 026 | 048 | 070 | 092 | 115 | 137 | 159 | 181 | 203 |
| 11.0 10.5 | 96 | 226 | 248 | 270 | 292 | 314 | 336 | 358 | 380 | 403 | 425 |
| 13.2 12.6 | | | | | | | | | | | |
| 15.4 14.7 | 97 | 447 | 469 | 491 | 513 | 535 | 557 | 579 | 601 | 623 | 645 |
| 17.6 16.8 | 98 | 667 | 688 | 710 | 732 | 754 | 776 | 798 | 820 | 842 | 863 |
| 19.8 18.9 | 99 | 29 885 | 907 | 929 | 951 | 973 | 994 | *016 | '038 | '060 | '081 |
| | 200 | 30 103 | 125 | 146 | 168 | 190 | 211 | 233 | 255 | 276 | 298 |

Prop. Parts

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 200 | 30 103 | 125 | 146 | 168 | 190 | 211 | 233 | 255 | 276 | 298 | |
| 01 | 320 | 341 | 363 | 384 | 406 | 428 | 449 | 471 | 492 | 514 | |
| 02 | 535 | 557 | 578 | 600 | 621 | 643 | 664 | 685 | 707 | 728 | p \| 22 21 |
| 03 | 750 | 771 | 792 | 814 | 835 | 856 | 878 | 899 | 920 | 942 | 1 \| 2.2 2.1 |
| | | | | | | | | | | | 2 \| 4.4 4.2 |
| 04 | 30 963 | 984 | *006 | *027 | *048 | *069 | *091 | *112 | *133 | *154 | 3 \| 6.6 6.3 |
| 05 | 31 175 | 197 | 218 | 239 | 260 | 281 | 302 | 323 | 345 | 366 | 4 \| 8.8 8.4 |
| 06 | 387 | 408 | 429 | 450 | 471 | 492 | 513 | 534 | 555 | 576 | 5 \| 11.0 10.5 |
| | | | | | | | | | | | 6 \| 13.2 12.6 |
| 07 | 597 | 618 | 639 | 660 | 681 | 702 | 723 | 744 | 765 | 785 | 7 \| 15.4 14.7 |
| 08 | 31 806 | 827 | 848 | 869 | 890 | 911 | 931 | 952 | 973 | 994 | 8 \| 17.6 16.8 |
| 09 | 32 015 | 035 | 056 | 077 | 098 | 118 | 139 | 160 | 181 | 201 | 9 \| 19.8 18.9 |
| 210 | 222 | 243 | 263 | 284 | 305 | 325 | 346 | 366 | 387 | 408 | |
| 11 | 428 | 449 | 469 | 490 | 510 | 531 | 552 | 572 | 593 | 613 | |
| 12 | 634 | 654 | 675 | 695 | 715 | 736 | 756 | 777 | 797 | 818 | 20 |
| 13 | 32 838 | 858 | 879 | 899 | 919 | 940 | 960 | 980 | *001 | *021 | 1 \| 2 |
| | | | | | | | | | | | 2 \| 4 |
| 14 | 33 041 | 062 | 082 | 102 | 122 | 143 | 163 | 183 | 203 | 224 | 3 \| 6 |
| 15 | 244 | 264 | 284 | 304 | 325 | 345 | 365 | 385 | 405 | 425 | 4 \| 8 |
| 16 | 445 | 465 | 486 | 506 | 526 | 546 | 566 | 586 | 606 | 626 | 5 \| 10 |
| | | | | | | | | | | | 6 \| 12 |
| 17 | 646 | 666 | 686 | 706 | 726 | 746 | 766 | 786 | 806 | 826 | 7 \| 14 |
| 18 | 33 846 | 866 | 885 | 905 | 925 | 945 | 965 | 985 | *005 | *025 | 8 \| 16 |
| 19 | 34 044 | 064 | 084 | 104 | 124 | 143 | 163 | 183 | 203 | 223 | 9 \| 18 |
| 220 | 242 | 262 | 282 | 301 | 321 | 341 | 361 | 380 | 400 | 420 | |
| 21 | 439 | 459 | 479 | 498 | 518 | 537 | 557 | 577 | 596 | 616 | |
| 22 | 635 | 655 | 674 | 694 | 713 | 733 | 753 | 772 | 792 | 811 | 19 |
| 23 | 34 830 | 850 | 869 | 889 | 908 | 928 | 947 | 967 | 986 | *005 | 1 \| 1.9 |
| | | | | | | | | | | | 2 \| 3.8 |
| 24 | 35 025 | 044 | 064 | 083 | 102 | 122 | 141 | 160 | 180 | 199 | 3 \| 5.7 |
| 25 | 218 | 238 | 257 | 276 | 295 | 315 | 334 | 353 | 372 | 392 | 4 \| 7.6 |
| 26 | 411 | 430 | 449 | 468 | 488 | 507 | 526 | 545 | 564 | 583 | 5 \| 9.5 |
| | | | | | | | | | | | 6 \| 11.4 |
| 27 | 603 | 622 | 641 | 660 | 679 | 698 | 717 | 736 | 755 | 774 | 7 \| 13.3 |
| 28 | 793 | 813 | 832 | 851 | 870 | 889 | 908 | 927 | 946 | 965 | 8 \| 15.2 |
| 29 | 35 984 | *003 | *021 | *040 | *059 | *078 | *097 | *116 | *135 | *154 | 9 \| 17.1 |
| 230 | 36 173 | 192 | 211 | 229 | 248 | 267 | 286 | 305 | 324 | 342 | |
| 31 | 361 | 380 | 399 | 418 | 436 | 455 | 474 | 493 | 511 | 530 | |
| 32 | 549 | 568 | 586 | 605 | 624 | 642 | 661 | 680 | 698 | 717 | 18 |
| 33 | 736 | 754 | 773 | 791 | 810 | 829 | 847 | 866 | 884 | 903 | 1 \| 1.8 |
| | | | | | | | | | | | 2 \| 3.6 |
| 34 | 36 922 | 940 | 959 | 977 | 996 | *014 | *033 | *051 | *070 | *088 | 3 \| 5.4 |
| 35 | 37 107 | 125 | 144 | 162 | 181 | 199 | 218 | 236 | 254 | 273 | 4 \| 7.2 |
| 36 | 291 | 310 | 328 | 346 | 365 | 383 | 401 | 420 | 438 | 457 | 5 \| 9.0 |
| | | | | | | | | | | | 6 \| 10.8 |
| 37 | 475 | 493 | 511 | 530 | 548 | 566 | 585 | 603 | 621 | 639 | 7 \| 12.6 |
| 38 | 658 | 676 | 694 | 712 | 731 | 749 | 767 | 785 | 803 | 822 | 8 \| 14.4 |
| 39 | 37 840 | 858 | 876 | 894 | 912 | 931 | 949 | 967 | 985 | *003 | 9 \| 16.2 |
| 240 | 38 021 | 039 | 057 | 075 | 093 | 112 | 130 | 148 | 166 | 184 | |
| 41 | 202 | 220 | 238 | 256 | 274 | 292 | 310 | 328 | 346 | 364 | |
| 42 | 382 | 399 | 417 | 435 | 453 | 471 | 489 | 507 | 525 | 543 | 17 |
| 43 | 561 | 578 | 596 | 614 | 632 | 650 | 668 | 686 | 703 | 721 | 1 \| 1.7 |
| | | | | | | | | | | | 2 \| 3.4 |
| 44 | 739 | 757 | 775 | 792 | 810 | 828 | 846 | 863 | 881 | 899 | 3 \| 5.1 |
| 45 | 38 917 | 934 | 952 | 970 | 987 | *005 | *023 | *041 | *058 | *076 | 4 \| 6.8 |
| 46 | 39 094 | 111 | 129 | 146 | 164 | 182 | 199 | 217 | 235 | 252 | 5 \| 8.5 |
| | | | | | | | | | | | 6 \| 10.2 |
| 47 | 270 | 287 | 305 | 322 | 340 | 358 | 375 | 393 | 410 | 428 | 7 \| 11.9 |
| 48 | 445 | 463 | 480 | 498 | 515 | 533 | 550 | 568 | 585 | 602 | 8 \| 13.6 |
| 49 | 620 | 637 | 655 | 672 | 690 | 707 | 724 | 742 | 759 | 777 | 9 \| 15.3 |
| 250 | 39 794 | 811 | 829 | 846 | 863 | 881 | 898 | 915 | 933 | 950 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 250 | 39 794 | 811 | 829 | 846 | 863 | 881 | 898 | 915 | 933 | 950 |
| | 51 | 39 967 | 985 | *002 | *019 | *037 | *054 | *071 | *088 | *106 | *123 |
| 18 | 52 | 40 140 | 157 | 175 | 192 | 209 | 226 | 243 | 261 | 278 | 295 |
| 1 1.8 | 53 | 312 | 329 | 346 | 364 | 381 | 398 | 415 | 432 | 449 | 466 |
| 2 3.6 | | | | | | | | | | | |
| 3 5.4 | 54 | 483 | 500 | 518 | 535 | 552 | 569 | 586 | 603 | 620 | 637 |
| 4 7.2 | 55 | 654 | 671 | 688 | 705 | 722 | 739 | 756 | 773 | 790 | 807 |
| 5 9.0 | 56 | 824 | 841 | 858 | 875 | 892 | 909 | 926 | 943 | 960 | 976 |
| 6 10.8 | | | | | | | | | | | |
| 7 12.6 | 57 | 40 993 | *010 | *027 | *044 | *061 | *078 | *095 | *111 | *128 | *145 |
| 8 14.4 | 58 | 41 162 | 179 | 196 | 212 | 229 | 246 | 263 | 280 | 296 | 313 |
| 9 16.2 | 59 | 330 | 347 | 363 | 380 | 397 | 414 | 430 | 447 | 464 | 481 |
| | 260 | 497 | 514 | 531 | 547 | 564 | 581 | 597 | 614 | 631 | 647 |
| | 61 | 664 | 681 | 697 | 714 | 731 | 747 | 764 | 780 | 797 | 814 |
| 17 | 62 | 830 | 847 | 863 | 880 | 896 | 913 | 929 | 946 | 963 | 979 |
| 1 1.7 | 63 | 41 996 | *012 | *029 | *045 | *062 | *078 | *095 | *111 | *127 | *144 |
| 2 3.4 | | | | | | | | | | | |
| 3 5.1 | 64 | 42 160 | 177 | 193 | 210 | 226 | 243 | 259 | 275 | 292 | 308 |
| 4 6.8 | 65 | 325 | 341 | 357 | 374 | 390 | 406 | 423 | 439 | 455 | 472 |
| 5 8.5 | 66 | 488 | 504 | 521 | 537 | 553 | 570 | 586 | 602 | 619 | 635 |
| 6 10.2 | | | | | | | | | | | |
| 7 11.9 | 67 | 651 | 667 | 684 | 700 | 716 | 732 | 749 | 765 | 781 | 797 |
| 8 13.6 | 68 | 813 | 830 | 846 | 862 | 878 | 894 | 911 | 927 | 943 | 959 |
| 9 15.3 | 69 | 42 975 | 991 | *008 | *024 | *040 | *056 | *072 | *088 | *104 | *120 |
| | 270 | 43 136 | 152 | 169 | 185 | 201 | 217 | 233 | 249 | 265 | 281 |
| | 71 | 297 | 313 | 329 | 345 | 361 | 377 | 393 | 409 | 425 | 441 |
| 16 | 72 | 457 | 473 | 489 | 505 | 521 | 537 | 553 | 569 | 584 | 600 |
| 1 1.6 | 73 | 616 | 632 | 648 | 664 | 680 | 696 | 712 | 727 | 743 | 759 |
| 2 3.2 | | | | | | | | | | | |
| 3 4.8 | 74 | 775 | 791 | 807 | 823 | 838 | 854 | 870 | 886 | 902 | 917 |
| 4 6.4 | 75 | 43 933 | 949 | 965 | 981 | 996 | *012 | *028 | *044 | *059 | *075 |
| 5 8.0 | 76 | 44 091 | 107 | 122 | 138 | 154 | 170 | 185 | 201 | 217 | 232 |
| 6 9.6 | | | | | | | | | | | |
| 7 11.2 | 77 | 248 | 264 | 279 | 295 | 311 | 326 | 342 | 358 | 373 | 389 |
| 8 12.8 | 78 | 404 | 420 | 436 | 451 | 467 | 483 | 498 | 514 | 529 | 545 |
| 9 14.4 | 79 | 560 | 576 | 592 | 607 | 623 | 638 | 654 | 669 | 685 | 700 |
| | 280 | 716 | 731 | 747 | 762 | 778 | 793 | 809 | 824 | 840 | 855 |
| | 81 | 44 871 | 886 | 902 | 917 | 932 | 948 | 963 | 979 | 994 | *010 |
| 15 | 82 | 45 025 | 040 | 056 | 071 | 086 | 102 | 117 | 133 | 148 | 163 |
| 1 1.5 | 83 | 179 | 194 | 209 | 225 | 240 | 255 | 271 | 286 | 301 | 317 |
| 2 3.0 | | | | | | | | | | | |
| 3 4.5 | 84 | 332 | 347 | 362 | 378 | 393 | 408 | 423 | 439 | 454 | 469 |
| 4 6.0 | 85 | 484 | 500 | 515 | 530 | 545 | 561 | 576 | 591 | 606 | 621 |
| 5 7.5 | 86 | 637 | 652 | 667 | 682 | 697 | 712 | 728 | 743 | 758 | 773 |
| 6 9.0 | | | | | | | | | | | |
| 7 10.5 | 87 | 788 | 803 | 818 | 834 | 849 | 864 | 879 | 894 | 909 | 924 |
| 8 12.0 | 88 | 45 939 | 954 | 969 | 984 | *000 | *015 | *030 | *045 | *060 | *075 |
| 9 13.5 | 89 | 46 090 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 210 | 225 |
| | 290 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 | 359 | 374 |
| | 91 | 389 | 404 | 419 | 434 | 449 | 464 | 479 | 494 | 509 | 523 |
| 14 | 92 | 538 | 553 | 568 | 583 | 598 | 613 | 627 | 642 | 657 | 672 |
| 1 1.4 | 93 | 687 | 702 | 716 | 731 | 746 | 761 | 776 | 790 | 805 | 820 |
| 2 2.8 | | | | | | | | | | | |
| 3 4.2 | 94 | 835 | 850 | 864 | 879 | 894 | 909 | 923 | 938 | 953 | 967 |
| 4 5.6 | 95 | 46 982 | 997 | *012 | *026 | *041 | *056 | *070 | *085 | *100 | *114 |
| 5 7.0 | 96 | 47 129 | 144 | 159 | 173 | 188 | 202 | 217 | 232 | 246 | 261 |
| 6 8.4 | | | | | | | | | | | |
| 7 9.8 | 97 | 276 | 290 | 305 | 319 | 334 | 349 | 363 | 378 | 392 | 407 |
| 8 11.2 | 98 | 422 | 436 | 451 | 465 | 480 | 494 | 509 | 524 | 538 | 553 |
| 9 12.6 | 99 | 567 | 582 | 596 | 611 | 625 | 640 | 654 | 669 | 683 | 698 |
| | 300 | 47 712 | 727 | 741 | 756 | 770 | 784 | 799 | 813 | 828 | 842 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

## Table II. Common Logarithms of Numbers to Five Decimal Places

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **300** | 47 712 | 727 | 741 | 756 | 770 | 784 | 799 | 813 | 828 | 842 |
| 01 | 47 857 | 871 | 885 | 900 | 914 | 929 | 943 | 958 | 972 | 986 |
| 02 | 48 001 | 015 | 029 | 044 | 058 | 073 | 087 | 101 | 116 | 130 |
| 03 | 144 | 159 | 173 | 187 | 202 | 216 | 230 | 244 | 259 | 273 |
| 04 | 287 | 302 | 316 | 330 | 344 | 359 | 373 | 387 | 401 | 416 |
| 05 | 430 | 444 | 458 | 473 | 487 | 501 | 515 | 530 | 544 | 558 |
| 06 | 572 | 586 | 601 | 615 | 629 | 643 | 657 | 671 | 686 | 700 |
| 07 | 714 | 728 | 742 | 756 | 770 | 785 | 799 | 813 | 827 | 841 |
| 08 | 855 | 869 | 883 | 897 | 911 | 926 | 940 | 954 | 968 | 982 |
| 09 | 48 996 | *010 | *024 | *038 | *052 | *066 | *080 | *094 | *108 | *122 |
| **310** | 49 136 | 150 | 164 | 178 | 192 | 206 | 220 | 234 | 248 | 262 |
| 11 | 276 | 290 | 304 | 318 | 332 | 346 | 360 | 374 | 388 | 402 |
| 12 | 415 | 429 | 443 | 457 | 471 | 485 | 499 | 513 | 527 | 541 |
| 13 | 554 | 568 | 582 | 596 | 610 | 624 | 638 | 651 | 665 | 679 |
| 14 | 693 | 707 | 721 | 734 | 748 | 762 | 776 | 790 | 803 | 817 |
| 15 | 831 | 845 | 859 | 872 | 886 | 900 | 914 | 927 | 941 | 955 |
| 16 | 49 969 | 982 | 996 | *010 | *024 | *037 | *051 | *065 | *079 | *092 |
| 17 | 50 106 | 120 | 133 | 147 | 161 | 174 | 188 | 202 | 215 | 229 |
| 18 | 243 | 256 | 270 | 284 | 297 | 311 | 325 | 338 | 352 | 365 |
| 19 | 379 | 393 | 406 | 420 | 433 | 447 | 461 | 474 | 488 | 501 |
| **320** | 515 | 529 | 542 | 556 | 569 | 583 | 596 | 610 | 623 | 637 |
| 21 | 651 | 664 | 678 | 691 | 705 | 718 | 732 | 745 | 759 | 772 |
| 22 | 786 | 799 | 813 | 826 | 840 | 853 | 866 | 880 | 893 | 907 |
| 23 | 50 920 | 934 | 947 | 961 | 974 | 987 | *001 | *014 | *028 | *041 |
| 24 | 51 055 | 068 | 081 | 095 | 108 | 121 | 135 | 148 | 162 | 175 |
| 25 | 188 | 202 | 215 | 228 | 242 | 255 | 268 | 282 | 295 | 308 |
| 26 | 322 | 335 | 348 | 362 | 375 | 388 | 402 | 415 | 428 | 441 |
| 27 | 455 | 468 | 481 | 495 | 508 | 521 | 534 | 548 | 561 | 574 |
| 28 | 587 | 601 | 614 | 627 | 640 | 654 | 667 | 680 | 693 | 706 |
| 29 | 720 | 733 | 746 | 759 | 772 | 786 | 799 | 812 | 825 | 838 |
| **330** | 851 | 865 | 878 | 891 | 904 | 917 | 930 | 943 | 957 | 970 |
| 31 | 51 983 | 996 | *009 | *022 | *035 | *048 | *061 | *075 | *088 | *101 |
| 32 | 52 114 | 127 | 140 | 153 | 166 | 179 | 192 | 205 | 218 | 231 |
| 33 | 244 | 257 | 270 | 284 | 297 | 310 | 323 | 336 | 349 | 362 |
| 34 | 375 | 388 | 401 | 414 | 427 | 440 | 453 | 466 | 479 | 492 |
| 35 | 504 | 517 | 530 | 543 | 556 | 569 | 582 | 595 | 608 | 621 |
| 36 | 634 | 647 | 660 | 673 | 686 | 699 | 711 | 724 | 737 | 750 |
| 37 | 763 | 776 | 789 | 802 | 815 | 827 | 840 | 853 | 866 | 879 |
| 38 | 52 892 | 905 | 917 | 930 | 943 | 956 | 969 | 982 | 994 | *007 |
| 39 | 53 020 | 033 | 046 | 058 | 071 | 084 | 097 | 110 | 122 | 135 |
| **340** | 148 | 161 | 173 | 186 | 199 | 212 | 224 | 237 | 250 | 263 |
| 41 | 275 | 288 | 301 | 314 | 326 | 339 | 352 | 364 | 377 | 390 |
| 42 | 403 | 415 | 428 | 441 | 453 | 466 | 479 | 491 | 504 | 517 |
| 43 | 529 | 542 | 555 | 567 | 580 | 593 | 605 | 618 | 631 | 643 |
| 44 | 656 | 668 | 681 | 694 | 706 | 719 | 732 | 744 | 757 | 769 |
| 45 | 782 | 794 | 807 | 820 | 832 | 845 | 857 | 870 | 882 | 895 |
| 46 | 53 908 | 920 | 933 | 945 | 958 | 970 | 983 | 995 | *008 | *020 |
| 47 | 54 033 | 045 | 058 | 070 | 083 | 095 | 108 | 120 | 133 | 145 |
| 48 | 158 | 170 | 183 | 195 | 208 | 220 | 233 | 245 | 258 | 270 |
| 49 | 283 | 295 | 307 | 320 | 332 | 345 | 357 | 370 | 382 | 394 |
| **350** | 54 407 | 419 | 432 | 444 | 456 | 469 | 481 | 494 | 506 | 518 |
| **N** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |

| Prop. Parts | |
|---|---|
| | 15 |
| 1 | 1.5 |
| 2 | 3.0 |
| 3 | 4.5 |
| 4 | 6.0 |
| 5 | 7.5 |
| 6 | 9.0 |
| 7 | 10.5 |
| 8 | 12.0 |
| 9 | 13.5 |

| Prop. Parts | |
|---|---|
| | 14 |
| 1 | 1.4 |
| 2 | 2.8 |
| 3 | 4.2 |
| 4 | 5.6 |
| 5 | 7.0 |
| 6 | 8.4 |
| 7 | 9.8 |
| 8 | 11.2 |
| 9 | 12.6 |

| Prop. Parts | |
|---|---|
| | 13 |
| 1 | 1.3 |
| 2 | 2.6 |
| 3 | 3.9 |
| 4 | 5.2 |
| 5 | 6.5 |
| 6 | 7.8 |
| 7 | 9.1 |
| 8 | 10.4 |
| 9 | 11.7 |

| Prop. Parts | |
|---|---|
| | 12 |
| 1 | 1.2 |
| 2 | 2.4 |
| 3 | 3.6 |
| 4 | 4.8 |
| 5 | 6.0 |
| 6 | 7.2 |
| 7 | 8.4 |
| 8 | 9.6 |
| 9 | 10.8 |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | **350** | 54 407 | 419 | 432 | 444 | 456 | 469 | 481 | 494 | 506 | 518 |
| | 51 | 531 | 543 | 555 | 568 | 580 | 593 | 605 | 617 | 630 | 642 |
| | 52 | 654 | 667 | 679 | 691 | 704 | 716 | 728 | 741 | 753 | 765 |
| | 53 | 777 | 790 | 802 | 814 | 827 | 839 | 851 | 864 | 876 | 888 |
| 13 | | | | | | | | | | | |
| 1 1.3 | 54 | 54 900 | 913 | 925 | 937 | 949 | 962 | 974 | 986 | 998 | *011 |
| 2 2.6 | 55 | 55 023 | 035 | 047 | 060 | 072 | 084 | 096 | 108 | 121 | 133 |
| 3 3.9 | 56 | 145 | 157 | 169 | 182 | 194 | 206 | 218 | 230 | 242 | 255 |
| 4 5.2 | | | | | | | | | | | |
| 5 6.5 | 57 | 267 | 279 | 291 | 303 | 315 | 328 | 340 | 352 | 364 | 376 |
| 6 7.8 | 58 | 388 | 400 | 413 | 425 | 437 | 449 | 461 | 473 | 485 | 497 |
| 7 9.1 | 59 | 509 | 522 | 534 | 546 | 558 | 570 | 582 | 594 | 606 | 618 |
| 8 10.4 | | | | | | | | | | | |
| 9 11.7 | **360** | 630 | 642 | 654 | 666 | 678 | 691 | 703 | 715 | 727 | 739 |
| | 61 | 751 | 763 | 775 | 787 | 799 | 811 | 823 | 835 | 847 | 859 |
| | 62 | 871 | 883 | 895 | 907 | 919 | 931 | 943 | 955 | 967 | 979 |
| | 63 | 55 991 | *003 | *015 | *027 | *038 | *050 | *062 | *074 | *086 | *098 |
| | 64 | 56 110 | 122 | 134 | 146 | 158 | 170 | 182 | 194 | 205 | 217 |
| | 65 | 229 | 241 | 253 | 265 | 277 | 289 | 301 | 312 | 324 | 336 |
| | 66 | 348 | 360 | 372 | 384 | 396 | 407 | 419 | 431 | 443 | 455 |
| 12 | | | | | | | | | | | |
| 1 1.2 | 67 | 467 | 478 | 490 | 502 | 514 | 526 | 538 | 549 | 561 | 573 |
| 2 2.4 | 68 | 585 | 597 | 608 | 620 | 632 | 644 | 656 | 667 | 679 | 691 |
| 3 3.6 | 69 | 703 | 714 | 726 | 738 | 750 | 761 | 773 | 785 | 797 | 808 |
| 4 4.8 | | | | | | | | | | | |
| 5 6.0 | **370** | 820 | 832 | 844 | 855 | 867 | 879 | 891 | 902 | 914 | 926 |
| 6 7.2 | | | | | | | | | | | |
| 7 8.4 | 71 | 56 937 | 949 | 961 | 972 | 984 | 996 | *008 | *019 | *031 | *043 |
| 8 9.6 | 72 | 57 054 | 066 | 078 | 089 | 101 | 113 | 124 | 136 | 148 | 159 |
| 9 10.8 | 73 | 171 | 183 | 194 | 206 | 217 | 229 | 241 | 252 | 264 | 276 |
| | 74 | 287 | 299 | 310 | 322 | 334 | 345 | 357 | 368 | 380 | 392 |
| | 75 | 403 | 415 | 426 | 438 | 449 | 461 | 473 | 484 | 496 | 507 |
| | 76 | 519 | 530 | 542 | 553 | 565 | 576 | 588 | 600 | 611 | 623 |
| | 77 | 634 | 646 | 657 | 669 | 680 | 692 | 703 | 715 | 726 | 738 |
| | 78 | 749 | 761 | 772 | 784 | 795 | 807 | 818 | 830 | 841 | 852 |
| | 79 | 864 | 875 | 887 | 898 | 910 | 921 | 933 | 944 | 955 | 967 |
| 11 | | | | | | | | | | | |
| 1 1.1 | **380** | 57 978 | 990 | *001 | *013 | *024 | *035 | *047 | *058 | *070 | *081 |
| 2 2.2 | | | | | | | | | | | |
| 3 3.3 | 81 | 58 092 | 104 | 115 | 127 | 138 | 149 | 161 | 172 | 184 | 195 |
| 4 4.4 | 82 | 206 | 218 | 229 | 240 | 252 | 263 | 274 | 286 | 297 | 309 |
| 5 5.5 | 83 | 320 | 331 | 343 | 354 | 365 | 377 | 388 | 399 | 410 | 422 |
| 6 6.6 | | | | | | | | | | | |
| 7 7.7 | 84 | 433 | 444 | 456 | 467 | 478 | 490 | 501 | 512 | 524 | 535 |
| 8 8.8 | 85 | 546 | 557 | 569 | 580 | 591 | 602 | 614 | 625 | 636 | 647 |
| 9 9.9 | 86 | 659 | 670 | 681 | 692 | 704 | 715 | 726 | 737 | 749 | 760 |
| | 87 | 771 | 782 | 794 | 805 | 816 | 827 | 838 | 850 | 861 | 872 |
| | 88 | 883 | 894 | 906 | 917 | 928 | 939 | 950 | 961 | 973 | 984 |
| | 89 | 58 995 | *006 | *017 | *028 | *040 | *051 | *062 | *073 | *084 | *095 |
| | **390** | 59 106 | 118 | 129 | 140 | 151 | 162 | 173 | 184 | 195 | 207 |
| 10 | | | | | | | | | | | |
| 1 1.0 | 91 | 218 | 229 | 240 | 251 | 262 | 273 | 284 | 295 | 306 | 318 |
| 2 2.0 | 92 | 329 | 340 | 351 | 362 | 373 | 384 | 395 | 406 | 417 | 428 |
| 3 3.0 | 93 | 439 | 450 | 461 | 472 | 483 | 494 | 506 | 517 | 528 | 539 |
| 4 4.0 | | | | | | | | | | | |
| 5 5.0 | 94 | 550 | 561 | 572 | 583 | 594 | 605 | 616 | 627 | 638 | 649 |
| 6 6.0 | 95 | 660 | 671 | 682 | 693 | 704 | 715 | 726 | 737 | 748 | 759 |
| 7 7.0 | 96 | 770 | 780 | 791 | 802 | 813 | 824 | 835 | 846 | 857 | 868 |
| 8 8.0 | | | | | | | | | | | |
| 9 9.0 | 97 | 879 | 890 | 901 | 912 | 923 | 934 | 945 | 956 | 966 | 977 |
| | 98 | 59 988 | 999 | *010 | *021 | *032 | *043 | *054 | *065 | *076 | *086 |
| | 99 | 60 097 | 108 | 119 | 130 | 141 | 152 | 163 | 173 | 184 | 195 |
| | **400** | 60 206 | 217 | 228 | 239 | 249 | 260 | 271 | 282 | 293 | 304 |
| **Prop. Parts** | **N** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |

## Table II. Common Logarithms of Numbers to Five Decimal Places

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 400 | 60 206 | 217 | 228 | 239 | 249 | 260 | 271 | 282 | 293 | 304 | |
| 01 | 314 | 325 | 336 | 347 | 358 | 369 | 379 | 390 | 401 | 412 | |
| 02 | 423 | 433 | 444 | 455 | 466 | 477 | 487 | 498 | 509 | 520 | |
| 03 | 531 | 541 | 552 | 563 | 574 | 584 | 595 | 606 | 617 | 627 | |
| 04 | 638 | 649 | 660 | 670 | 681 | 692 | 703 | 713 | 724 | 736 | |
| 05 | 746 | 756 | 767 | 778 | 788 | 799 | 810 | 821 | 831 | 842 | |
| 06 | 853 | 863 | 874 | 885 | 895 | 906 | 917 | 927 | 938 | 949 | |
| 07 | 60 959 | 970 | 981 | 991 | *002 | *013 | *023 | *034 | *045 | *055 | 11 / 1 1.1 / 2 2.2 / 3 3.3 / 4 4.4 / 5 5.5 / 6 6.6 / 7 7.7 / 8 8.8 / 9 9.9 |
| 08 | 61 066 | 077 | 087 | 098 | 109 | 119 | 130 | 140 | 151 | 162 | |
| 09 | 172 | 183 | 194 | 204 | 215 | 225 | 236 | 247 | 257 | 268 | |
| 410 | 278 | 289 | 300 | 310 | 321 | 331 | 342 | 352 | 363 | 374 | |
| 11 | 384 | 395 | 405 | 416 | 426 | 437 | 448 | 458 | 469 | 479 | |
| 12 | 490 | 500 | 511 | 521 | 532 | 542 | 553 | 563 | 574 | 584 | |
| 13 | 595 | 606 | 616 | 627 | 637 | 648 | 658 | 669 | 679 | 690 | |
| 14 | 700 | 711 | 721 | 731 | 742 | 752 | 763 | 773 | 784 | 794 | |
| 15 | 805 | 815 | 826 | 836 | 847 | 857 | 868 | 878 | 888 | 899 | |
| 16 | 61 909 | 920 | 930 | 941 | 951 | 962 | 972 | 982 | 993 | *003 | |
| 17 | 62 014 | 024 | 034 | 045 | 055 | 066 | 076 | 086 | 097 | 107 | |
| 18 | 118 | 128 | 138 | 149 | 159 | 170 | 180 | 190 | 201 | 211 | |
| 19 | 221 | 232 | 242 | 252 | 263 | 273 | 284 | 294 | 304 | 315 | |
| 420 | 325 | 335 | 346 | 356 | 366 | 377 | 387 | 397 | 408 | 418 | |
| 21 | 428 | 439 | 449 | 459 | 469 | 480 | 490 | 500 | 511 | 521 | |
| 22 | 531 | 542 | 552 | 562 | 572 | 583 | 593 | 603 | 613 | 624 | |
| 23 | 634 | 644 | 655 | 665 | 675 | 685 | 696 | 706 | 716 | 726 | 10 / 1 1.0 / 2 2.0 / 3 3.0 / 4 4.0 / 5 5.0 / 6 6.0 / 7 7.0 / 8 8.0 / 9 9.0 |
| 24 | 737 | 747 | 757 | 767 | 778 | 788 | 798 | 808 | 818 | 829 | |
| 25 | 839 | 849 | 859 | 870 | 880 | 890 | 900 | 910 | 921 | 931 | |
| 26 | 62 941 | 951 | 961 | 972 | 982 | 992 | *002 | *012 | *022 | *033 | |
| 27 | 63 043 | 053 | 063 | 073 | 083 | 094 | 104 | 114 | 124 | 134 | |
| 28 | 144 | 155 | 165 | 175 | 185 | 195 | 205 | 215 | 225 | 236 | |
| 29 | 246 | 256 | 266 | 276 | 286 | 296 | 306 | 317 | 327 | 337 | |
| 430 | 347 | 357 | 367 | 377 | 387 | 397 | 407 | 417 | 428 | 438 | |
| 31 | 448 | 458 | 468 | 478 | 488 | 498 | 508 | 518 | 528 | 538 | |
| 32 | 548 | 558 | 568 | 579 | 589 | 599 | 609 | 619 | 629 | 639 | |
| 33 | 649 | 659 | 669 | 679 | 689 | 699 | 709 | 719 | 729 | 739 | |
| 34 | 749 | 759 | 769 | 779 | 789 | 799 | 809 | 819 | 829 | 839 | |
| 35 | 849 | 859 | 869 | 879 | 889 | 899 | 909 | 919 | 929 | 939 | |
| 36 | 63 949 | 959 | 969 | 979 | 988 | 998 | *008 | *018 | *028 | *038 | |
| 37 | 64 048 | 058 | 068 | 078 | 088 | 098 | 108 | 118 | 128 | 137 | |
| 38 | 147 | 157 | 167 | 177 | 187 | 197 | 207 | 217 | 227 | 237 | 9 / 1 0.9 / 2 1.8 / 3 2.7 / 4 3.6 / 5 4.5 / 6 5.4 / 7 6.3 / 8 7.2 / 9 8.1 |
| 39 | 246 | 256 | 266 | 276 | 286 | 296 | 306 | 316 | 326 | 335 | |
| 440 | 345 | 355 | 365 | 375 | 385 | 395 | 404 | 414 | 424 | 434 | |
| 41 | 444 | 454 | 464 | 473 | 483 | 493 | 503 | 513 | 523 | 532 | |
| 42 | 542 | 552 | 562 | 572 | 582 | 591 | 601 | 611 | 621 | 631 | |
| 43 | 640 | 650 | 660 | 670 | 680 | 689 | 699 | 709 | 719 | 729 | |
| 44 | 738 | 748 | 758 | 768 | 777 | 787 | 797 | 807 | 816 | 826 | |
| 45 | 836 | 846 | 856 | 865 | 875 | 885 | 895 | 904 | 914 | 924 | |
| 46 | 64 933 | 943 | 953 | 963 | 972 | 982 | 992 | *002 | *011 | *021 | |
| 47 | 65 031 | 040 | 050 | 060 | 070 | 079 | 089 | 099 | 108 | 118 | |
| 48 | 128 | 137 | 147 | 157 | 167 | 176 | 186 | 196 | 205 | 215 | |
| 49 | 225 | 234 | 244 | 254 | 263 | 273 | 283 | 292 | 302 | 312 | |
| 450 | 65 321 | 331 | 341 | 350 | 360 | 369 | 379 | 389 | 398 | 408 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 450 | 65 321 | 331 | 341 | 350 | 360 | 369 | 379 | 389 | 398 | 408 |
| | 51 | 418 | 427 | 437 | 447 | 456 | 466 | 475 | 485 | 495 | 504 |
| | 52 | 514 | 523 | 533 | 543 | 552 | 562 | 571 | 581 | 591 | 600 |
| | 53 | 610 | 619 | 629 | 639 | 648 | 658 | 667 | 677 | 686 | 696 |
| | 54 | 706 | 715 | 725 | 734 | 744 | 753 | 763 | 772 | 782 | 792 |
| | 55 | 801 | 811 | 820 | 830 | 839 | 849 | 858 | 868 | 877 | 887 |
| | 56 | 896 | 906 | 916 | 925 | 935 | 944 | 954 | 963 | 973 | 982 |
| 10 | | | | | | | | | | | |
| 1 1.0 | 57 | 65 992 | *001 | *011 | *020 | *030 | *039 | *049 | *058 | *068 | *077 |
| 2 2.0 | 58 | 66 087 | 096 | 106 | 115 | 124 | 134 | 143 | 153 | 162 | 172 |
| 3 3.0 | 59 | 181 | 191 | 200 | 210 | 219 | 229 | 238 | 247 | 257 | 266 |
| 4 4.0 | | | | | | | | | | | |
| 5 5.0 | 460 | 276 | 285 | 295 | 304 | 314 | 323 | 332 | 342 | 351 | 361 |
| 6 6.0 | | | | | | | | | | | |
| 7 7.0 | 61 | 370 | 380 | 389 | 398 | 408 | 417 | 427 | 436 | 445 | 455 |
| 8 8.0 | 62 | 464 | 474 | 483 | 492 | 502 | 511 | 521 | 530 | 539 | 549 |
| 9 9.0 | 63 | 558 | 567 | 577 | 586 | 596 | 605 | 614 | 624 | 633 | 642 |
| | 64 | 652 | 661 | 671 | 680 | 689 | 699 | 708 | 717 | 727 | 736 |
| | 65 | 745 | 755 | 764 | 773 | 783 | 792 | 801 | 811 | 820 | 829 |
| | 66 | 839 | 848 | 857 | 867 | 876 | 885 | 894 | 904 | 913 | 922 |
| | 67 | 66 932 | 941 | 950 | 960 | 969 | 978 | 987 | 997 | *006 | *015 |
| | 68 | 67 025 | 034 | 043 | 052 | 062 | 071 | 080 | 089 | 099 | 108 |
| | 69 | 117 | 127 | 136 | 145 | 154 | 164 | 173 | 182 | 191 | 201 |
| | 470 | 210 | 219 | 228 | 237 | 247 | 256 | 265 | 274 | 284 | 293 |
| | 71 | 302 | 311 | 321 | 330 | 339 | 348 | 357 | 367 | 376 | 385 |
| 9 | 72 | 394 | 403 | 413 | 422 | 431 | 440 | 449 | 459 | 468 | 477 |
| 1 0.9 | 73 | 486 | 495 | 504 | 514 | 523 | 532 | 541 | 550 | 560 | 569 |
| 2 1.8 | | | | | | | | | | | |
| 3 2.7 | 74 | 578 | 587 | 596 | 605 | 614 | 624 | 633 | 642 | 651 | 660 |
| 4 3.6 | 75 | 669 | 679 | 688 | 697 | 706 | 715 | 724 | 733 | 742 | 752 |
| 5 4.5 | 76 | 761 | 770 | 779 | 788 | 797 | 806 | 815 | 825 | 834 | 843 |
| 6 5.4 | | | | | | | | | | | |
| 7 6.3 | 77 | 852 | 861 | 870 | 879 | 888 | 897 | 906 | 916 | 925 | 934 |
| 8 7.2 | 78 | 67 943 | 952 | 961 | 970 | 979 | 988 | 997 | *006 | *015 | *024 |
| 9 8.1 | 79 | 68 034 | 043 | 052 | 061 | 070 | 079 | 088 | 097 | 106 | 115 |
| | 480 | 124 | 133 | 142 | 151 | 160 | 169 | 178 | 187 | 196 | 205 |
| | 81 | 215 | 224 | 233 | 242 | 251 | 260 | 269 | 278 | 287 | 296 |
| | 82 | 305 | 314 | 323 | 332 | 341 | 350 | 359 | 368 | 377 | 386 |
| | 83 | 395 | 404 | 413 | 422 | 431 | 440 | 449 | 458 | 467 | 476 |
| | 84 | 485 | 494 | 502 | 511 | 520 | 529 | 538 | 547 | 556 | 565 |
| | 85 | 574 | 583 | 592 | 601 | 610 | 619 | 628 | 637 | 646 | 655 |
| | 86 | 664 | 673 | 681 | 690 | 699 | 708 | 717 | 726 | 735 | 744 |
| | 87 | 753 | 762 | 771 | 780 | 789 | 797 | 806 | 815 | 824 | 833 |
| 8 | 88 | 842 | 851 | 860 | 869 | 878 | 886 | 895 | 904 | 913 | 922 |
| 1 0.8 | 89 | 68 931 | 940 | 949 | 958 | 966 | 975 | 984 | 993 | *002 | *011 |
| 2 1.6 | 490 | 69 020 | 028 | 037 | 046 | 055 | 064 | 073 | 082 | 090 | 099 |
| 3 2.4 | | | | | | | | | | | |
| 4 3.2 | 91 | 108 | 117 | 126 | 135 | 144 | 152 | 161 | 170 | 179 | 188 |
| 5 4.0 | 92 | 197 | 205 | 214 | 223 | 232 | 241 | 249 | 258 | 267 | 276 |
| 6 4.8 | 93 | 285 | 294 | 302 | 311 | 320 | 329 | 338 | 346 | 355 | 364 |
| 7 5.6 | | | | | | | | | | | |
| 8 6.4 | 94 | 373 | 381 | 390 | 399 | 408 | 417 | 425 | 434 | 443 | 452 |
| 9 7.2 | 95 | 461 | 469 | 478 | 487 | 496 | 504 | 513 | 522 | 531 | 539 |
| | 96 | 548 | 557 | 566 | 574 | 583 | 592 | 601 | 609 | 618 | 627 |
| | 97 | 636 | 644 | 653 | 662 | 671 | 679 | 688 | 697 | 705 | 714 |
| | 98 | 723 | 732 | 740 | 749 | 758 | 767 | 775 | 784 | 793 | 801 |
| | 99 | 810 | 819 | 827 | 836 | 845 | 854 | 862 | 871 | 880 | 888 |
| | 500 | 69 897 | 906 | 914 | 923 | 932 | 940 | 949 | 958 | 966 | 975 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 500 | 69 897 | 906 | 914 | 923 | 932 | 940 | 949 | 958 | 966 | 975 | |
| 01 | 69 984 | 992 | *001 | *010 | *018 | *027 | *036 | *044 | *053 | *062 | |
| 02 | 70 070 | 079 | 088 | 096 | 105 | 114 | 122 | 131 | 140 | 148 | |
| 03 | 157 | 165 | 174 | 183 | 191 | 200 | 209 | 217 | 226 | 234 | |
| 04 | 243 | 252 | 260 | 269 | 278 | 286 | 295 | 303 | 312 | 321 | |
| 05 | 329 | 338 | 346 | 355 | 364 | 372 | 381 | 389 | 398 | 406 | |
| 06 | 415 | 424 | 432 | 441 | 449 | 458 | 467 | 475 | 484 | 492 | |
| 07 | 501 | 509 | 518 | 526 | 535 | 544 | 552 | 561 | 569 | 578 | 9<br>1 0.9<br>2 1.8<br>3 2.7<br>4 3.6<br>5 4.5<br>6 5.4<br>7 6.3<br>8 7.2<br>9 8.1 |
| 08 | 586 | 595 | 603 | 612 | 621 | 629 | 638 | 646 | 655 | 663 | |
| 09 | 672 | 680 | 689 | 697 | 706 | 714 | 723 | 731 | 740 | 749 | |
| 510 | 757 | 766 | 774 | 783 | 791 | 800 | 808 | 817 | 825 | 834 | |
| 11 | 842 | 851 | 859 | 868 | 876 | 885 | 893 | 902 | 910 | 919 | |
| 12 | 70 927 | 935 | 944 | 952 | 961 | 969 | 978 | 986 | 995 | *003 | |
| 13 | 71 012 | 020 | 029 | 037 | 046 | 054 | 063 | 071 | 079 | 088 | |
| 14 | 096 | 105 | 113 | 122 | 130 | 139 | 147 | 155 | 164 | 172 | |
| 15 | 181 | 189 | 198 | 206 | 214 | 223 | 231 | 240 | 248 | 257 | |
| 16 | 265 | 273 | 282 | 290 | 299 | 307 | 315 | 324 | 332 | 341 | |
| 17 | 349 | 357 | 366 | 374 | 383 | 391 | 399 | 408 | 416 | 425 | |
| 18 | 433 | 441 | 450 | 458 | 466 | 475 | 483 | 492 | 500 | 508 | |
| 19 | 517 | 525 | 533 | 542 | 550 | 559 | 567 | 575 | 584 | 592 | |
| 520 | 600 | 609 | 617 | 625 | 634 | 642 | 650 | 659 | 667 | 675 | |
| 21 | 684 | 692 | 700 | 709 | 717 | 725 | 734 | 742 | 750 | 759 | |
| 22 | 767 | 775 | 784 | 792 | 800 | 809 | 817 | 825 | 834 | 842 | |
| 23 | 850 | 858 | 867 | 875 | 883 | 892 | 900 | 908 | 917 | 925 | 8<br>1 0.8<br>2 1.6<br>3 2.4<br>4 3.2<br>5 4.0<br>6 4.8<br>7 5.6<br>8 6.4<br>9 7.2 |
| 24 | 71 933 | 941 | 950 | 958 | 966 | 975 | 983 | 991 | 999 | *008 | |
| 25 | 72 016 | 024 | 032 | 041 | 049 | 057 | 066 | 074 | 082 | 090 | |
| 26 | 099 | 107 | 115 | 123 | 132 | 140 | 148 | 156 | 165 | 173 | |
| 27 | 181 | 189 | 198 | 206 | 214 | 222 | 230 | 239 | 247 | 255 | |
| 28 | 263 | 272 | 280 | 288 | 296 | 304 | 313 | 321 | 329 | 337 | |
| 29 | 346 | 354 | 362 | 370 | 378 | 387 | 395 | 403 | 411 | 419 | |
| 530 | 428 | 436 | 444 | 452 | 460 | 469 | 477 | 485 | 493 | 501 | |
| 31 | 509 | 518 | 526 | 534 | 542 | 550 | 558 | 567 | 575 | 583 | |
| 32 | 591 | 599 | 607 | 616 | 624 | 632 | 640 | 648 | 656 | 665 | |
| 33 | 673 | 681 | 689 | 697 | 705 | 713 | 722 | 730 | 738 | 746 | |
| 34 | 754 | 762 | 770 | 779 | 787 | 795 | 803 | 811 | 819 | 827 | |
| 35 | 835 | 843 | 852 | 860 | 868 | 876 | 884 | 892 | 900 | 908 | |
| 36 | 916 | 925 | 933 | 941 | 949 | 957 | 965 | 973 | 981 | 989 | |
| 37 | 72 997 | *006 | *014 | *022 | *030 | *038 | *046 | *054 | *062 | *070 | |
| 38 | 73 078 | 086 | 094 | 102 | 111 | 119 | 127 | 135 | 143 | 151 | 7<br>1 0.7<br>2 1.4<br>3 2.1<br>4 2.8<br>5 3.5<br>6 4.2<br>7 4.9<br>8 5.6<br>9 6.3 |
| 39 | 159 | 167 | 175 | 183 | 191 | 199 | 207 | 215 | 223 | 231 | |
| 540 | 239 | 247 | 255 | 263 | 272 | 280 | 288 | 296 | 304 | 312 | |
| 41 | 320 | 328 | 336 | 344 | 352 | 360 | 368 | 376 | 384 | 392 | |
| 42 | 400 | 408 | 416 | 424 | 432 | 440 | 448 | 456 | 464 | 472 | |
| 43 | 480 | 488 | 496 | 504 | 512 | 520 | 528 | 536 | 544 | 552 | |
| 44 | 560 | 568 | 576 | 584 | 592 | 600 | 608 | 616 | 624 | 632 | |
| 45 | 640 | 648 | 656 | 664 | 672 | 679 | 687 | 695 | 703 | 711 | |
| 46 | 719 | 727 | 735 | 743 | 751 | 759 | 767 | 775 | 783 | 791 | |
| 47 | 799 | 807 | 815 | 823 | 830 | 838 | 846 | 854 | 862 | 870 | |
| 48 | 878 | 886 | 894 | 902 | 910 | 918 | 926 | 933 | 941 | 949 | |
| 49 | 73 957 | 965 | 973 | 981 | 989 | 997 | *005 | *013 | *020 | *028 | |
| 550 | 74 036 | 044 | 052 | 060 | 068 | 076 | 084 | 092 | 099 | 107 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 550 | 74 036 | 044 | 052 | 060 | 068 | 076 | 084 | 092 | 099 | 107 |
| | 51 | 115 | 123 | 131 | 139 | 147 | 155 | 162 | 170 | 178 | 186 |
| | 52 | 194 | 202 | 210 | 218 | 225 | 233 | 241 | 249 | 257 | 265 |
| | 53 | 273 | 280 | 288 | 296 | 304 | 312 | 320 | 327 | 335 | 343 |
| | 54 | 351 | 359 | 367 | 374 | 382 | 390 | 398 | 406 | 414 | 421 |
| | 55 | 429 | 437 | 445 | 453 | 461 | 468 | 476 | 484 | 492 | 500 |
| | 56 | 507 | 515 | 523 | 531 | 539 | 547 | 554 | 562 | 570 | 578 |
| | 57 | 586 | 593 | 601 | 609 | 617 | 624 | 632 | 640 | 648 | 656 |
| | 58 | 663 | 671 | 679 | 687 | 695 | 702 | 710 | 718 | 726 | 733 |
| | 59 | 741 | 749 | 757 | 764 | 772 | 780 | 788 | 796 | 803 | 811 |
| | 560 | 819 | 827 | 834 | 842 | 850 | 858 | 865 | 873 | 881 | 889 |
| | 61 | 896 | 904 | 912 | 920 | 927 | 935 | 943 | 950 | 958 | 966 |
| 8 | 62 | 74 974 | 981 | 989 | 997 | *005 | *012 | *020 | *028 | *035 | *043 |
| 1 0.8 | 63 | 75 051 | 059 | 066 | 074 | 082 | 089 | 097 | 105 | 113 | 120 |
| 2 1.6 | | | | | | | | | | | |
| 3 2.4 | 64 | 128 | 136 | 143 | 151 | 159 | 166 | 174 | 182 | 189 | 197 |
| 4 3.2 | 65 | 205 | 213 | 220 | 228 | 236 | 243 | 251 | 259 | 266 | 274 |
| 5 4.0 | 66 | 282 | 289 | 297 | 305 | 312 | 320 | 328 | 335 | 343 | 351 |
| 6 4.8 | | | | | | | | | | | |
| 7 5.6 | 67 | 358 | 366 | 374 | 381 | 389 | 397 | 404 | 412 | 420 | 427 |
| 8 6.4 | 68 | 435 | 442 | 450 | 458 | 465 | 473 | 481 | 488 | 496 | 504 |
| 9 7.2 | 69 | 511 | 519 | 526 | 534 | 542 | 549 | 557 | 565 | 572 | 580 |
| | 570 | 587 | 595 | 603 | 610 | 618 | 626 | 633 | 641 | 648 | 656 |
| | 71 | 664 | 671 | 679 | 686 | 694 | 702 | 709 | 717 | 724 | 732 |
| | 72 | 740 | 747 | 755 | 762 | 770 | 778 | 785 | 793 | 800 | 808 |
| | 73 | 815 | 823 | 831 | 838 | 846 | 853 | 861 | 868 | 876 | 884 |
| | 74 | 891 | 899 | 906 | 914 | 921 | 929 | 937 | 944 | 952 | 959 |
| | 75 | 75 967 | 974 | 982 | 989 | 997 | *005 | *012 | *020 | *027 | *035 |
| | 76 | 76 042 | 050 | 057 | 065 | 072 | 080 | 087 | 095 | 103 | 110 |
| | 77 | 118 | 125 | 133 | 140 | 148 | 155 | 163 | 170 | 178 | 185 |
| | 78 | 193 | 200 | 208 | 215 | 223 | 230 | 238 | 245 | 253 | 260 |
| | 79 | 268 | 275 | 283 | 290 | 298 | 305 | 313 | 320 | 328 | 335 |
| | 580 | 343 | 350 | 358 | 365 | 373 | 380 | 388 | 395 | 403 | 410 |
| | 81 | 418 | 425 | 433 | 440 | 448 | 455 | 462 | 470 | 477 | 485 |
| 7 | 82 | 492 | 500 | 507 | 515 | 522 | 530 | 537 | 545 | 552 | 559 |
| 1 0.7 | 83 | 567 | 574 | 582 | 589 | 597 | 604 | 612 | 619 | 626 | 634 |
| 2 1.4 | | | | | | | | | | | |
| 3 2.1 | 84 | 641 | 649 | 656 | 664 | 671 | 678 | 686 | 693 | 701 | 708 |
| 4 2.8 | 85 | 716 | 723 | 730 | 738 | 745 | 753 | 760 | 768 | 775 | 782 |
| 5 3.5 | 86 | 790 | 797 | 805 | 812 | 819 | 827 | 834 | 842 | 849 | 856 |
| 6 4.2 | | | | | | | | | | | |
| 7 4.9 | 87 | 864 | 871 | 879 | 886 | 893 | 901 | 908 | 916 | 923 | 930 |
| 8 5.6 | 88 | 76 938 | 945 | 953 | 960 | 967 | 975 | 982 | 989 | 997 | *004 |
| 9 6.3 | 89 | 77 012 | 019 | 026 | 034 | 041 | 048 | 056 | 063 | 070 | 078 |
| | 590 | 085 | 093 | 100 | 107 | 115 | 122 | 129 | 137 | 144 | 151 |
| | 91 | 159 | 166 | 173 | 181 | 188 | 195 | 203 | 210 | 217 | 225 |
| | 92 | 232 | 240 | 247 | 254 | 262 | 269 | 276 | 283 | 291 | 298 |
| | 93 | 305 | 313 | 320 | 327 | 335 | 342 | 349 | 357 | 364 | 371 |
| | 94 | 379 | 386 | 393 | 401 | 408 | 415 | 422 | 430 | 437 | 444 |
| | 95 | 452 | 459 | 466 | 474 | 481 | 488 | 495 | 503 | 510 | 517 |
| | 96 | 525 | 532 | 539 | 546 | 554 | 561 | 568 | 576 | 583 | 590 |
| | 97 | 597 | 605 | 612 | 619 | 627 | 634 | 641 | 648 | 656 | 663 |
| | 98 | 670 | 677 | 685 | 692 | 699 | 706 | 714 | 721 | 728 | 735 |
| | 99 | 743 | 750 | 757 | 764 | 772 | 779 | 786 | 793 | 801 | 808 |
| | 600 | 77 815 | 822 | 830 | 837 | 844 | 851 | 859 | 866 | 873 | 880 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 600 | 77 815 | 822 | 830 | 837 | 844 | 851 | 859 | 866 | 873 | 880 | |
| 01 | 887 | 895 | 902 | 909 | 916 | 924 | 931 | 938 | 945 | 952 | |
| 02 | 77 960 | 967 | 974 | 981 | 988 | 996 | *003 | *010 | *017 | *025 | |
| 03 | 78 032 | 039 | 046 | 053 | 061 | 068 | 075 | 082 | 089 | 097 | |
| 04 | 104 | 111 | 118 | 125 | 132 | 140 | 147 | 154 | 161 | 168 | |
| 05 | 176 | 183 | 190 | 197 | 204 | 211 | 219 | 226 | 233 | 240 | |
| 06 | 247 | 254 | 262 | 269 | 276 | 283 | 290 | 297 | 305 | 312 | |
| 07 | 319 | 326 | 333 | 340 | 347 | 355 | 362 | 369 | 376 | 383 | 8 \| 1 0.8 \| 2 1.6 \| 3 2.4 \| 4 3.2 \| 5 4.0 \| 6 4.8 \| 7 5.6 \| 8 6.4 \| 9 7.2 |
| 08 | 390 | 398 | 405 | 412 | 419 | 426 | 433 | 440 | 447 | 455 | |
| 09 | 462 | 469 | 476 | 483 | 490 | 497 | 504 | 512 | 519 | 526 | |
| 610 | 533 | 540 | 547 | 554 | 561 | 569 | 576 | 583 | 590 | 597 | |
| 11 | 604 | 611 | 618 | 625 | 633 | 640 | 647 | 654 | 661 | 668 | |
| 12 | 675 | 682 | 689 | 696 | 704 | 711 | 718 | 725 | 732 | 739 | |
| 13 | 746 | 753 | 760 | 767 | 774 | 781 | 789 | 796 | 803 | 810 | |
| 14 | 817 | 824 | 831 | 838 | 845 | 852 | 859 | 866 | 873 | 880 | |
| 15 | 888 | 895 | 902 | 909 | 916 | 923 | 930 | 937 | 944 | 951 | |
| 16 | 78 958 | 965 | 972 | 979 | 986 | 993 | *000 | *007 | *014 | *021 | |
| 17 | 79 029 | 036 | 043 | 050 | 057 | 064 | 071 | 078 | 085 | 092 | |
| 18 | 099 | 106 | 113 | 120 | 127 | 134 | 141 | 148 | 155 | 162 | |
| 19 | 169 | 176 | 183 | 190 | 197 | 204 | 211 | 218 | 225 | 232 | |
| 620 | 239 | 246 | 253 | 260 | 267 | 274 | 281 | 288 | 295 | 302 | |
| 21 | 309 | 316 | 323 | 330 | 337 | 344 | 351 | 358 | 365 | 372 | |
| 22 | 379 | 386 | 393 | 400 | 407 | 414 | 421 | 428 | 435 | 442 | 7 \| 1 0.7 \| 2 1.4 \| 3 2.1 \| 4 2.8 \| 5 3.5 \| 6 4.2 \| 7 4.9 \| 8 5.6 \| 9 6.3 |
| 23 | 449 | 456 | 463 | 470 | 477 | 484 | 491 | 498 | 505 | 511 | |
| 24 | 518 | 525 | 532 | 539 | 546 | 553 | 560 | 567 | 574 | 581 | |
| 25 | 588 | 595 | 602 | 609 | 616 | 623 | 630 | 637 | 644 | 650 | |
| 26 | 657 | 664 | 671 | 678 | 685 | 692 | 699 | 706 | 713 | 720 | |
| 27 | 727 | 734 | 741 | 748 | 754 | 761 | 768 | 775 | 782 | 789 | |
| 28 | 796 | 803 | 810 | 817 | 824 | 831 | 837 | 844 | 851 | 858 | |
| 29 | 865 | 872 | 879 | 886 | 893 | 900 | 906 | 913 | 920 | 927 | |
| 630 | 79 934 | 941 | 948 | 955 | 962 | 969 | 975 | 982 | 989 | 996 | |
| 31 | 80 003 | 010 | 017 | 024 | 030 | 037 | 044 | 051 | 058 | 065 | |
| 32 | 072 | 079 | 085 | 092 | 099 | 106 | 113 | 120 | 127 | 134 | |
| 33 | 140 | 147 | 154 | 161 | 168 | 175 | 182 | 188 | 195 | 202 | |
| 34 | 209 | 216 | 223 | 229 | 236 | 243 | 250 | 257 | 264 | 271 | |
| 35 | 277 | 284 | 291 | 298 | 305 | 312 | 318 | 325 | 332 | 339 | |
| 36 | 346 | 353 | 359 | 366 | 373 | 380 | 387 | 393 | 400 | 407 | |
| 37 | 414 | 421 | 428 | 434 | 441 | 448 | 455 | 462 | 468 | 475 | |
| 38 | 482 | 489 | 496 | 502 | 509 | 516 | 523 | 530 | 536 | 543 | 6 \| 1 0.6 \| 2 1.2 \| 3 1.8 \| 4 2.4 \| 5 3.0 \| 6 3.6 \| 7 4.2 \| 8 4.8 \| 9 5.4 |
| 39 | 550 | 557 | 564 | 570 | 577 | 584 | 591 | 598 | 604 | 611 | |
| 640 | 618 | 625 | 632 | 638 | 645 | 652 | 659 | 665 | 672 | 679 | |
| 41 | 686 | 693 | 699 | 706 | 713 | 720 | 726 | 733 | 740 | 747 | |
| 42 | 754 | 760 | 767 | 774 | 781 | 787 | 794 | 801 | 808 | 814 | |
| 43 | 821 | 828 | 835 | 841 | 848 | 855 | 862 | 868 | 875 | 882 | |
| 44 | 889 | 895 | 902 | 909 | 916 | 922 | 929 | 936 | 943 | 949 | |
| 45 | 80 956 | 963 | 969 | 976 | 983 | 990 | 996 | *003 | *010 | *017 | |
| 46 | 81 023 | 030 | 037 | 043 | 050 | 057 | 064 | 070 | 077 | 084 | |
| 47 | 090 | 097 | 104 | 111 | 117 | 124 | 131 | 137 | 144 | 151 | |
| 48 | 158 | 164 | 171 | 178 | 184 | 191 | 198 | 204 | 211 | 218 | |
| 49 | 224 | 231 | 238 | 245 | 251 | 258 | 265 | 271 | 278 | 285 | |
| 650 | 81 291 | 298 | 305 | 311 | 318 | 325 | 331 | 338 | 345 | 351 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 650 | 81 291 | 298 | 305 | 311 | 318 | 325 | 331 | 338 | 345 | 351 |
| | 51 | 358 | 365 | 371 | 378 | 385 | 391 | 398 | 405 | 411 | 418 |
| | 52 | 425 | 431 | 438 | 445 | 451 | 458 | 465 | 471 | 478 | 485 |
| | 53 | 491 | 498 | 505 | 511 | 518 | 525 | 531 | 538 | 544 | 551 |
| | 54 | 558 | 564 | 571 | 578 | 584 | 591 | 598 | 604 | 611 | 617 |
| | 55 | 624 | 631 | 637 | 644 | 651 | 657 | 664 | 671 | 677 | 684 |
| | 56 | 690 | 697 | 704 | 710 | 717 | 723 | 730 | 737 | 743 | 750 |
| | 57 | 757 | 763 | 770 | 776 | 783 | 790 | 796 | 803 | 809 | 816 |
| | 58 | 823 | 829 | 836 | 842 | 849 | 856 | 862 | 869 | 875 | 882 |
| | 59 | 889 | 895 | 902 | 908 | 915 | 921 | 928 | 935 | 941 | 948 |
| | 660 | 81 954 | 961 | 968 | 974 | 981 | 987 | 994 | *000 | *007 | *014 |
| | 61 | 82 020 | 027 | 033 | 040 | 046 | 053 | 060 | 066 | 073 | 079 |
| | 62 | 086 | 092 | 099 | 105 | 112 | 119 | 125 | 132 | 138 | 145 |
| 7 | 63 | 151 | 158 | 164 | 171 | 178 | 184 | 191 | 197 | 204 | 210 |
| 1 0.7 | | | | | | | | | | | |
| 2 1.4 | | | | | | | | | | | |
| 3 2.1 | 64 | 217 | 223 | 230 | 236 | 243 | 249 | 256 | 263 | 269 | 276 |
| 4 2.8 | 65 | 282 | 289 | 295 | 302 | 308 | 315 | 321 | 328 | 334 | 341 |
| 5 3.5 | 66 | 347 | 354 | 360 | 367 | 373 | 380 | 387 | 393 | 400 | 406 |
| 6 4.2 | | | | | | | | | | | |
| 7 4.9 | 67 | 413 | 419 | 426 | 432 | 439 | 445 | 452 | 458 | 465 | 471 |
| 8 5.6 | 68 | 478 | 484 | 491 | 497 | 504 | 510 | 517 | 523 | 530 | 536 |
| 9 6.3 | 69 | 543 | 549 | 556 | 562 | 569 | 575 | 582 | 588 | 595 | 601 |
| | 670 | 607 | 614 | 620 | 627 | 633 | 640 | 646 | 653 | 659 | 666 |
| | 71 | 672 | 679 | 685 | 692 | 698 | 705 | 711 | 718 | 724 | 730 |
| | 72 | 737 | 743 | 750 | 756 | 763 | 769 | 776 | 782 | 789 | 795 |
| | 73 | 802 | 808 | 814 | 821 | 827 | 834 | 840 | 847 | 853 | 860 |
| | 74 | 866 | 872 | 879 | 885 | 892 | 898 | 905 | 911 | 918 | 924 |
| | 75 | 930 | 937 | 943 | 950 | 956 | 963 | 969 | 975 | 982 | 988 |
| | 76 | 82 995 | *001 | *008 | *014 | *020 | *027 | *033 | *040 | *046 | *052 |
| | 77 | 83 059 | 065 | 072 | 078 | 085 | 091 | 097 | 104 | 110 | 117 |
| | 78 | 123 | 129 | 136 | 142 | 149 | 155 | 161 | 168 | 174 | 181 |
| | 79 | 187 | 193 | 200 | 206 | 213 | 219 | 225 | 232 | 238 | 245 |
| | 680 | 251 | 257 | 264 | 270 | 276 | 283 | 289 | 296 | 302 | 308 |
| | 81 | 315 | 321 | 327 | 334 | 340 | 347 | 353 | 359 | 366 | 372 |
| | 82 | 378 | 385 | 391 | 398 | 404 | 410 | 417 | 423 | 429 | 436 |
| 6 | 83 | 442 | 448 | 455 | 461 | 467 | 474 | 480 | 487 | 493 | 499 |
| 1 0.6 | | | | | | | | | | | |
| 2 1.2 | | | | | | | | | | | |
| 3 1.8 | | | | | | | | | | | |
| 4 2.4 | 84 | 506 | 512 | 518 | 525 | 531 | 537 | 544 | 550 | 556 | 563 |
| 5 3.0 | 85 | 569 | 575 | 582 | 588 | 594 | 601 | 607 | 613 | 620 | 626 |
| 6 3.6 | 86 | 632 | 639 | 645 | 651 | 658 | 664 | 670 | 677 | 683 | 689 |
| 7 4.2 | | | | | | | | | | | |
| 8 4.8 | 87 | 696 | 702 | 708 | 715 | 721 | 727 | 734 | 740 | 746 | 753 |
| 9 5.4 | 88 | 759 | 765 | 771 | 778 | 784 | 790 | 797 | 803 | 809 | 816 |
| | 89 | 822 | 828 | 835 | 841 | 847 | 853 | 860 | 866 | 872 | 879 |
| | 690 | 885 | 891 | 897 | 904 | 910 | 916 | 923 | 929 | 935 | 942 |
| | 91 | 83 948 | 954 | 960 | 967 | 973 | 979 | 985 | 992 | 998 | *004 |
| | 92 | 84 011 | 017 | 023 | 029 | 036 | 042 | 048 | 055 | 061 | 067 |
| | 93 | 073 | 080 | 086 | 092 | 098 | 105 | 111 | 117 | 123 | 130 |
| | 94 | 136 | 142 | 148 | 155 | 161 | 167 | 173 | 180 | 186 | 192 |
| | 95 | 198 | 205 | 211 | 217 | 223 | 230 | 236 | 242 | 248 | 255 |
| | 96 | 261 | 267 | 273 | 280 | 286 | 292 | 298 | 305 | 311 | 317 |
| | 97 | 323 | 330 | 336 | 342 | 348 | 354 | 361 | 367 | 373 | 379 |
| | 98 | 386 | 392 | 398 | 404 | 410 | 417 | 423 | 429 | 435 | 442 |
| | 99 | 448 | 454 | 460 | 466 | 473 | 479 | 485 | 491 | 497 | 504 |
| | 700 | 84 510 | 516 | 522 | 528 | 535 | 541 | 547 | 553 | 559 | 566 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 700 | 84 510 | 516 | 522 | 528 | 535 | 541 | 547 | 553 | 559 | 566 | |
| 01 | 572 | 578 | 584 | 590 | 597 | 603 | 609 | 615 | 621 | 628 | |
| 02 | 634 | 640 | 646 | 652 | 658 | 665 | 671 | 677 | 683 | 689 | |
| 03 | 696 | 702 | 708 | 714 | 720 | 726 | 733 | 739 | 745 | 751 | |
| 04 | 757 | 763 | 770 | 776 | 782 | 788 | 794 | 800 | 807 | 813 | |
| 05 | 819 | 825 | 831 | 837 | 844 | 850 | 856 | 862 | 868 | 874 | |
| 06 | 880 | 887 | 893 | 899 | 905 | 911 | 917 | 924 | 930 | 936 | |
| 07 | 84 942 | 948 | 954 | 960 | 967 | 973 | 979 | 985 | 991 | 997 | 7 |
| 08 | 85 003 | 009 | 016 | 022 | 028 | 034 | 040 | 046 | 052 | 058 | 1 0.7 |
| 09 | 065 | 071 | 077 | 083 | 089 | 095 | 101 | 107 | 114 | 120 | 2 1.4 |
| 710 | 126 | 132 | 138 | 144 | 150 | 156 | 163 | 169 | 175 | 181 | 3 2.1 |
| 11 | 187 | 193 | 199 | 205 | 211 | 217 | 224 | 230 | 236 | 242 | 4 2.8 |
| 12 | 248 | 254 | 260 | 266 | 272 | 278 | 285 | 291 | 297 | 303 | 5 3.5 |
| 13 | 309 | 315 | 321 | 327 | 333 | 339 | 345 | 352 | 358 | 364 | 6 4.2 |
| 14 | 370 | 376 | 382 | 388 | 394 | 400 | 406 | 412 | 418 | 425 | 7 4.9 |
| 15 | 431 | 437 | 443 | 449 | 455 | 461 | 467 | 473 | 479 | 485 | 8 5.6 |
| 16 | 491 | 497 | 503 | 509 | 516 | 522 | 528 | 534 | 540 | 546 | 9 6.3 |
| 17 | 552 | 558 | 564 | 570 | 576 | 582 | 588 | 594 | 600 | 606 | |
| 18 | 612 | 618 | 625 | 631 | 637 | 643 | 649 | 655 | 661 | 667 | |
| 19 | 673 | 679 | 685 | 691 | 697 | 703 | 709 | 715 | 721 | 727 | |
| 720 | 733 | 739 | 745 | 751 | 757 | 763 | 769 | 775 | 781 | 788 | |
| 21 | 794 | 800 | 806 | 812 | 818 | 824 | 830 | 836 | 842 | 848 | |
| 22 | 854 | 860 | 866 | 872 | 878 | 884 | 890 | 896 | 902 | 908 | |
| 23 | 914 | 920 | 926 | 932 | 938 | 944 | 950 | 956 | 962 | 968 | 6 |
| 24 | 85 974 | 980 | 986 | 992 | 998 | *004 | *010 | *016 | *022 | *028 | 1 0.6 |
| 25 | 86 034 | 040 | 046 | 052 | 058 | 064 | 070 | 076 | 082 | 088 | 2 1.2 |
| 26 | 094 | 100 | 106 | 112 | 118 | 124 | 130 | 136 | 141 | 147 | 3 1.8 |
| 27 | 153 | 159 | 165 | 171 | 177 | 183 | 189 | 195 | 201 | 207 | 4 2.4 |
| 28 | 213 | 219 | 225 | 231 | 237 | 243 | 249 | 255 | 261 | 267 | 5 3.0 |
| 29 | 273 | 279 | 285 | 291 | 297 | 303 | 308 | 314 | 320 | 326 | 6 3.6 |
| 730 | 332 | 338 | 344 | 350 | 356 | 362 | 368 | 374 | 380 | 386 | 7 4.2 |
| 31 | 392 | 398 | 404 | 410 | 415 | 421 | 427 | 433 | 439 | 445 | 8 4.8 |
| 32 | 451 | 457 | 463 | 469 | 475 | 481 | 487 | 493 | 499 | 504 | 9 5.4 |
| 33 | 510 | 516 | 522 | 528 | 534 | 540 | 546 | 552 | 558 | 564 | |
| 34 | 570 | 576 | 581 | 587 | 593 | 599 | 605 | 611 | 617 | 623 | |
| 35 | 629 | 635 | 641 | 646 | 652 | 658 | 664 | 670 | 676 | 682 | |
| 36 | 688 | 694 | 700 | 705 | 711 | 717 | 723 | 729 | 735 | 741 | |
| 37 | 747 | 753 | 759 | 764 | 770 | 776 | 782 | 788 | 794 | 800 | |
| 38 | 806 | 812 | 817 | 823 | 829 | 835 | 841 | 847 | 853 | 859 | 5 |
| 39 | 864 | 870 | 876 | 882 | 888 | 894 | 900 | 906 | 911 | 917 | 1 0.5 |
| 740 | 923 | 929 | 935 | 941 | 947 | 953 | 958 | 964 | 970 | 976 | 2 1.0 |
| 41 | 86 982 | 988 | 994 | 999 | *005 | *011 | *017 | *023 | *029 | *035 | 3 1.5 |
| 42 | 87 040 | 046 | 052 | 058 | 064 | 070 | 075 | 081 | 087 | 093 | 4 2.0 |
| 43 | 099 | 105 | 111 | 116 | 122 | 128 | 134 | 140 | 146 | 151 | 5 2.5 |
| 44 | 157 | 163 | 169 | 175 | 181 | 186 | 192 | 198 | 204 | 210 | 6 3.0 |
| 45 | 216 | 221 | 227 | 233 | 239 | 245 | 251 | 256 | 262 | 268 | 7 3.5 |
| 46 | 274 | 280 | 286 | 291 | 297 | 303 | 309 | 315 | 320 | 326 | 8 4.0 |
| 47 | 332 | 338 | 344 | 349 | 355 | 361 | 367 | 373 | 379 | 384 | 9 4.5 |
| 48 | 390 | 396 | 402 | 408 | 413 | 419 | 425 | 431 | 437 | 442 | |
| 49 | 448 | 454 | 460 | 466 | 471 | 477 | 483 | 489 | 495 | 500 | |
| 750 | 87 506 | 512 | 518 | 523 | 529 | 535 | 541 | 547 | 552 | 558 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

## Table II. Common Logarithms of Numbers to Five Decimal Places

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 750 | 87 506 | 512 | 518 | 523 | 529 | 535 | 541 | 547 | 552 | 558 |
| | 51 | 564 | 570 | 576 | 581 | 587 | 593 | 599 | 604 | 610 | 616 |
| | 52 | 622 | 628 | 633 | 639 | 645 | 651 | 656 | 662 | 668 | 674 |
| | 53 | 679 | 685 | 691 | 697 | 703 | 708 | 714 | 720 | 726 | 731 |
| | 54 | 737 | 743 | 749 | 754 | 760 | 766 | 772 | 777 | 783 | 789 |
| | 55 | 795 | 800 | 806 | 812 | 818 | 823 | 829 | 835 | 841 | 846 |
| | 56 | 852 | 858 | 864 | 869 | 875 | 881 | 887 | 892 | 898 | 904 |
| | 57 | 910 | 915 | 921 | 927 | 933 | 938 | 944 | 950 | 955 | 961 |
| | 58 | 87 967 | 973 | 978 | 984 | 990 | 996 | *001 | *007 | *013 | *018 |
| | 59 | 88 024 | 030 | 036 | 041 | 047 | 053 | 058 | 064 | 070 | 076 |
| | 760 | 081 | 087 | 093 | 098 | 104 | 110 | 116 | 121 | 127 | 133 |
| | 61 | 138 | 144 | 150 | 156 | 161 | 167 | 173 | 178 | 184 | 190 |
| | 62 | 195 | 201 | 207 | 213 | 218 | 224 | 230 | 235 | 241 | 247 |
| 6 | 63 | 252 | 258 | 264 | 270 | 275 | 281 | 287 | 292 | 298 | 304 |
| 1 0.6 | | | | | | | | | | | |
| 2 1.2 | | | | | | | | | | | |
| 3 1.8 | 64 | 309 | 315 | 321 | 326 | 332 | 338 | 343 | 349 | 355 | 360 |
| 4 2.4 | 65 | 366 | 372 | 377 | 383 | 389 | 395 | 400 | 406 | 412 | 417 |
| 5 3.0 | 66 | 423 | 429 | 434 | 440 | 446 | 451 | 457 | 463 | 468 | 474 |
| 6 3.6 | | | | | | | | | | | |
| 7 4.2 | 67 | 480 | 485 | 491 | 497 | 502 | 508 | 513 | 519 | 525 | 530 |
| 8 4.8 | 68 | 536 | 542 | 547 | 553 | 559 | 564 | 570 | 576 | 581 | 587 |
| 9 5.4 | 69 | 593 | 598 | 604 | 610 | 615 | 621 | 627 | 632 | 638 | 643 |
| | 770 | 649 | 655 | 660 | 666 | 672 | 677 | 683 | 689 | 694 | 700 |
| | 71 | 705 | 711 | 717 | 722 | 728 | 734 | 739 | 745 | 750 | 756 |
| | 72 | 762 | 767 | 773 | 779 | 784 | 790 | 795 | 801 | 807 | 812 |
| | 73 | 818 | 824 | 829 | 835 | 840 | 846 | 852 | 857 | 863 | 868 |
| | 74 | 874 | 880 | 885 | 891 | 897 | 902 | 908 | 913 | 919 | 925 |
| | 75 | 930 | 936 | 941 | 947 | 953 | 958 | 964 | 969 | 975 | 981 |
| | 76 | 88 986 | 992 | 997 | *003 | *009 | *014 | *020 | *025 | *031 | *037 |
| | 77 | 89 042 | 048 | 053 | 059 | 064 | 070 | 076 | 081 | 087 | 092 |
| | 78 | 098 | 104 | 109 | 115 | 120 | 126 | 131 | 137 | 143 | 148 |
| | 79 | 154 | 159 | 165 | 170 | 176 | 182 | 187 | 193 | 198 | 204 |
| | 780 | 209 | 215 | 221 | 226 | 232 | 237 | 243 | 248 | 254 | 260 |
| | 81 | 265 | 271 | 276 | 282 | 287 | 293 | 298 | 304 | 310 | 315 |
| | 82 | 321 | 326 | 332 | 337 | 343 | 348 | 354 | 360 | 365 | 371 |
| 5 | 83 | 376 | 382 | 387 | 393 | 398 | 404 | 409 | 415 | 421 | 426 |
| 1 0.5 | | | | | | | | | | | |
| 2 1.0 | | | | | | | | | | | |
| 3 1.5 | | | | | | | | | | | |
| 4 2.0 | 84 | 432 | 437 | 443 | 448 | 454 | 459 | 465 | 470 | 476 | 481 |
| 5 2.5 | 85 | 487 | 492 | 498 | 504 | 509 | 515 | 520 | 526 | 531 | 537 |
| 6 3.0 | 86 | 542 | 548 | 553 | 559 | 564 | 570 | 575 | 581 | 586 | 592 |
| 7 3.5 | | | | | | | | | | | |
| 8 4.0 | 87 | 597 | 603 | 609 | 614 | 620 | 625 | 631 | 636 | 642 | 647 |
| 9 4.5 | 88 | 653 | 658 | 664 | 669 | 675 | 680 | 686 | 691 | 697 | 702 |
| | 89 | 708 | 713 | 719 | 724 | 730 | 735 | 741 | 746 | 752 | 757 |
| | 790 | 763 | 768 | 774 | 779 | 785 | 790 | 796 | 801 | 807 | 812 |
| | 91 | 818 | 823 | 829 | 834 | 840 | 845 | 851 | 856 | 862 | 867 |
| | 92 | 873 | 878 | 883 | 889 | 894 | 900 | 905 | 911 | 916 | 922 |
| | 93 | 927 | 933 | 938 | 944 | 949 | 955 | 960 | 966 | 971 | 977 |
| | 94 | 89 982 | 988 | 993 | 998 | *004 | *009 | *015 | *020 | *026 | *031 |
| | 95 | 90 037 | 042 | 048 | 053 | 059 | 064 | 069 | 075 | 080 | 086 |
| | 96 | 091 | 097 | 102 | 108 | 113 | 119 | 124 | 129 | 135 | 140 |
| | 97 | 146 | 151 | 157 | 162 | 168 | 173 | 179 | 184 | 189 | 195 |
| | 98 | 200 | 206 | 211 | 217 | 222 | 227 | 233 | 238 | 244 | 249 |
| | 99 | 255 | 260 | 266 | 271 | 276 | 282 | 287 | 293 | 298 | 304 |
| | 800 | 90 309 | 314 | 320 | 325 | 331 | 336 | 342 | 347 | 352 | 358 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 800 | 90 309 | 314 | 320 | 325 | 331 | 336 | 342 | 347 | 352 | 358 | |
| 01 | 363 | 369 | 374 | 380 | 385 | 390 | 396 | 401 | 407 | 412 | |
| 02 | 417 | 423 | 428 | 434 | 439 | 445 | 450 | 455 | 461 | 466 | |
| 03 | 472 | 477 | 482 | 488 | 493 | 499 | 504 | 509 | 515 | 520 | |
| 04 | 526 | 531 | 536 | 542 | 547 | 553 | 558 | 563 | 569 | 574 | |
| 05 | 580 | 585 | 590 | 596 | 601 | 607 | 612 | 617 | 623 | 628 | |
| 06 | 634 | 639 | 644 | 650 | 655 | 660 | 666 | 671 | 677 | 682 | |
| 07 | 687 | 693 | 698 | 703 | 709 | 714 | 720 | 725 | 730 | 736 | |
| 08 | 741 | 747 | 752 | 757 | 763 | 768 | 773 | 779 | 784 | 789 | |
| 09 | 795 | 800 | 806 | 811 | 816 | 822 | 827 | 832 | 838 | 843 | |
| 810 | 849 | 854 | 859 | 865 | 870 | 875 | 881 | 886 | 891 | 897 | |
| 11 | 902 | 907 | 913 | 918 | 924 | 929 | 934 | 940 | 945 | 950 | |
| 12 | 90 956 | 961 | 966 | 972 | 977 | 982 | 988 | 993 | 998 | *004 | |
| 13 | 91 009 | 014 | 020 | 025 | 030 | 036 | 041 | 046 | 052 | 057 | 6<br>1 0.6<br>2 1.2 |
| 14 | 062 | 068 | 073 | 078 | 084 | 089 | 094 | 100 | 105 | 110 | 3 1.8 |
| 15 | 116 | 121 | 126 | 132 | 137 | 142 | 148 | 153 | 158 | 164 | 4 2.4 |
| 16 | 169 | 174 | 180 | 185 | 190 | 196 | 201 | 206 | 212 | 217 | 5 3.0<br>6 3.6 |
| 17 | 222 | 228 | 233 | 238 | 243 | 249 | 254 | 259 | 265 | 270 | 7 4.2 |
| 18 | 275 | 281 | 286 | 291 | 297 | 302 | 307 | 312 | 318 | 323 | 8 4.8 |
| 19 | 328 | 334 | 339 | 344 | 350 | 355 | 360 | 365 | 371 | 376 | 9 5.4 |
| 820 | 381 | 387 | 392 | 397 | 403 | 408 | 413 | 418 | 424 | 429 | |
| 21 | 434 | 440 | 445 | 450 | 455 | 461 | 466 | 471 | 477 | 482 | |
| 22 | 487 | 492 | 498 | 503 | 508 | 514 | 519 | 524 | 529 | 535 | |
| 23 | 540 | 545 | 551 | 556 | 561 | 566 | 572 | 577 | 582 | 587 | |
| 24 | 593 | 598 | 603 | 609 | 614 | 619 | 624 | 630 | 635 | 640 | |
| 25 | 645 | 651 | 656 | 661 | 666 | 672 | 677 | 682 | 687 | 693 | |
| 26 | 698 | 703 | 709 | 714 | 719 | 724 | 730 | 735 | 740 | 745 | |
| 27 | 751 | 756 | 761 | 766 | 772 | 777 | 782 | 787 | 793 | 798 | |
| 28 | 803 | 808 | 814 | 819 | 824 | 829 | 834 | 840 | 845 | 850 | |
| 29 | 855 | 861 | 866 | 871 | 876 | 882 | 887 | 892 | 897 | 903 | |
| 830 | 908 | 913 | 918 | 924 | 929 | 934 | 939 | 944 | 950 | 955 | |
| 31 | 91 960 | 965 | 971 | 976 | 981 | 986 | 991 | 997 | *002 | *007 | |
| 32 | 92 012 | 018 | 023 | 028 | 033 | 038 | 044 | 049 | 054 | 059 | 5<br>1 0.5 |
| 33 | 065 | 070 | 075 | 080 | 085 | 091 | 096 | 101 | 106 | 111 | 2 1.0<br>3 1.5 |
| 34 | 117 | 122 | 127 | 132 | 137 | 143 | 148 | 153 | 158 | 163 | 4 2.0 |
| 35 | 169 | 174 | 179 | 184 | 189 | 195 | 200 | 205 | 210 | 215 | 5 2.5 |
| 36 | 221 | 226 | 231 | 236 | 241 | 247 | 252 | 257 | 262 | 267 | 6 3.0<br>7 3.5 |
| 37 | 273 | 278 | 283 | 288 | 293 | 298 | 304 | 309 | 314 | 319 | 8 4.0 |
| 38 | 324 | 330 | 335 | 340 | 345 | 350 | 355 | 361 | 366 | 371 | 9 4.5 |
| 39 | 376 | 381 | 387 | 392 | 397 | 402 | 407 | 412 | 418 | 423 | |
| 840 | 428 | 433 | 438 | 443 | 449 | 454 | 459 | 464 | 469 | 474 | |
| 41 | 480 | 485 | 490 | 495 | 500 | 505 | 511 | 516 | 521 | 526 | |
| 42 | 531 | 536 | 542 | 547 | 552 | 557 | 562 | 567 | 572 | 578 | |
| 43 | 583 | 588 | 593 | 598 | 603 | 609 | 614 | 619 | 624 | 629 | |
| 44 | 634 | 639 | 645 | 650 | 655 | 660 | 665 | 670 | 675 | 681 | |
| 45 | 686 | 691 | 696 | 701 | 706 | 711 | 716 | 722 | 727 | 732 | |
| 46 | 737 | 742 | 747 | 752 | 758 | 763 | 768 | 773 | 778 | 783 | |
| 47 | 788 | 793 | 799 | 804 | 809 | 814 | 819 | 824 | 829 | 834 | |
| 48 | 840 | 845 | 850 | 855 | 860 | 865 | 870 | 875 | 881 | 886 | |
| 49 | 891 | 896 | 901 | 906 | 911 | 916 | 921 | 927 | 932 | 937 | |
| 850 | 92 942 | 947 | 952 | 957 | 962 | 967 | 973 | 978 | 983 | 988 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 850 | 92 942 | 947 | 952 | 957 | 962 | 967 | 973 | 978 | 983 | 988 |
| | 51 | 92 993 | 998 | *003 | *008 | *013 | *018 | *024 | *029 | *034 | *039 |
| | 52 | 93 044 | 049 | 054 | 059 | 064 | 069 | 075 | 080 | 085 | 090 |
| | 53 | 095 | 100 | 105 | 110 | 115 | 120 | 125 | 131 | 136 | 141 |
| | 54 | 146 | 151 | 156 | 161 | 166 | 171 | 176 | 181 | 186 | 192 |
| | 55 | 197 | 202 | 207 | 212 | 217 | 222 | 227 | 232 | 237 | 242 |
| | 56 | 247 | 252 | 258 | 263 | 268 | 273 | 278 | 283 | 288 | 293 |
| 6 | | | | | | | | | | | |
| 1 0.6 | 57 | 298 | 303 | 308 | 313 | 318 | 323 | 328 | 334 | 339 | 344 |
| 2 1.2 | 58 | 349 | 354 | 359 | 364 | 369 | 374 | 379 | 384 | 389 | 394 |
| 3 1.8 | 59 | 399 | 404 | 409 | 414 | 420 | 425 | 430 | 435 | 440 | 445 |
| 4 2.4 | | | | | | | | | | | |
| 5 3.0 | 860 | 450 | 455 | 460 | 465 | 470 | 475 | 480 | 485 | 490 | 495 |
| 6 3.6 | | | | | | | | | | | |
| 7 4.2 | 61 | 500 | 505 | 510 | 515 | 520 | 526 | 531 | 536 | 541 | 546 |
| 8 4.8 | 62 | 551 | 556 | 561 | 566 | 571 | 576 | 581 | 586 | 591 | 596 |
| 9 5.4 | 63 | 601 | 606 | 611 | 616 | 621 | 626 | 631 | 636 | 641 | 646 |
| | 64 | 651 | 656 | 661 | 666 | 671 | 676 | 682 | 687 | 692 | 697 |
| | 65 | 702 | 707 | 712 | 717 | 722 | 727 | 732 | 737 | 742 | 747 |
| | 66 | 752 | 757 | 762 | 767 | 772 | 777 | 782 | 787 | 792 | 797 |
| | 67 | 802 | 807 | 812 | 817 | 822 | 827 | 832 | 837 | 842 | 847 |
| | 68 | 852 | 857 | 862 | 867 | 872 | 877 | 882 | 887 | 892 | 897 |
| | 69 | 902 | 907 | 912 | 917 | 922 | 927 | 932 | 937 | 942 | 947 |
| | 870 | 93 952 | 957 | 962 | 967 | 972 | 977 | 982 | 987 | 992 | 997 |
| | 71 | 94 002 | 007 | 012 | 017 | 022 | 027 | 032 | 037 | 042 | 047 |
| | 72 | 052 | 057 | 062 | 067 | 072 | 077 | 082 | 086 | 091 | 096 |
| 5 | 73 | 101 | 106 | 111 | 116 | 121 | 126 | 131 | 136 | 141 | 146 |
| 1 0.5 | | | | | | | | | | | |
| 2 1.0 | 74 | 151 | 156 | 161 | 166 | 171 | 176 | 181 | 186 | 191 | 196 |
| 3 1.5 | 75 | 201 | 206 | 211 | 216 | 221 | 226 | 231 | 236 | 240 | 245 |
| 4 2.0 | 76 | 250 | 255 | 260 | 265 | 270 | 275 | 280 | 285 | 290 | 295 |
| 5 2.5 | | | | | | | | | | | |
| 6 3.0 | 77 | 300 | 305 | 310 | 315 | 320 | 325 | 330 | 335 | 340 | 345 |
| 7 3.5 | 78 | 349 | 354 | 359 | 364 | 369 | 374 | 379 | 384 | 389 | 394 |
| 8 4.0 | 79 | 399 | 404 | 409 | 414 | 419 | 424 | 429 | 433 | 438 | 443 |
| 9 4.5 | | | | | | | | | | | |
| | 880 | 448 | 453 | 458 | 463 | 468 | 473 | 478 | 483 | 488 | 493 |
| | 81 | 498 | 503 | 507 | 512 | 517 | 522 | 527 | 532 | 537 | 542 |
| | 82 | 547 | 552 | 557 | 562 | 567 | 571 | 576 | 581 | 586 | 591 |
| | 83 | 596 | 601 | 606 | 611 | 616 | 621 | 626 | 630 | 635 | 640 |
| | 84 | 645 | 650 | 655 | 660 | 665 | 670 | 675 | 680 | 685 | 689 |
| | 85 | 694 | 699 | 704 | 709 | 714 | 719 | 724 | 729 | 734 | 738 |
| | 86 | 743 | 748 | 753 | 758 | 763 | 768 | 773 | 778 | 783 | 787 |
| | 87 | 792 | 797 | 802 | 807 | 812 | 817 | 822 | 827 | 832 | 836 |
| | 88 | 841 | 846 | 851 | 856 | 861 | 866 | 871 | 876 | 880 | 885 |
| 4 | 89 | 890 | 895 | 900 | 905 | 910 | 915 | 919 | 924 | 929 | 934 |
| 1 0.4 | | | | | | | | | | | |
| 2 0.8 | 890 | 939 | 944 | 949 | 954 | 959 | 963 | 968 | 973 | 978 | 983 |
| 3 1.2 | | | | | | | | | | | |
| 4 1.6 | 91 | 94 988 | 993 | 998 | *002 | *007 | *012 | *017 | *022 | *027 | *032 |
| 5 2.0 | 92 | 95 036 | 041 | 046 | 051 | 056 | 061 | 066 | 071 | 075 | 080 |
| 6 2.4 | 93 | 085 | 090 | 095 | 100 | 105 | 109 | 114 | 119 | 124 | 129 |
| 7 2.8 | | | | | | | | | | | |
| 8 3.2 | 94 | 134 | 139 | 143 | 148 | 153 | 158 | 163 | 168 | 173 | 177 |
| 9 3.6 | 95 | 182 | 187 | 192 | 197 | 202 | 207 | 211 | 216 | 221 | 226 |
| | 96 | 231 | 236 | 240 | 245 | 250 | 255 | 260 | 265 | 270 | 274 |
| | 97 | 279 | 284 | 289 | 294 | 299 | 303 | 308 | 313 | 318 | 323 |
| | 98 | 328 | 332 | 337 | 342 | 347 | 352 | 357 | 361 | 366 | 371 |
| | 99 | 376 | 381 | 386 | 390 | 395 | 400 | 405 | 410 | 415 | 419 |
| | 900 | 95 424 | 429 | 434 | 439 | 444 | 448 | 453 | 458 | 463 | 468 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 900 | 95 424 | 429 | 434 | 439 | 444 | 448 | 453 | 458 | 463 | 468 | |
| 01 | 472 | 477 | 482 | 487 | 492 | 497 | 501 | 506 | 511 | 516 | |
| 02 | 521 | 525 | 530 | 535 | 540 | 545 | 550 | 554 | 559 | 564 | |
| 03 | 569 | 574 | 578 | 583 | 588 | 593 | 598 | 602 | 607 | 612 | |
| 04 | 617 | 622 | 626 | 631 | 636 | 641 | 646 | 650 | 655 | 660 | |
| 05 | 665 | 670 | 674 | 679 | 684 | 689 | 694 | 698 | 703 | 708 | |
| 06 | 713 | 718 | 722 | 727 | 732 | 737 | 742 | 746 | 751 | 756 | |
| 07 | 761 | 766 | 770 | 775 | 780 | 785 | 789 | 794 | 799 | 804 | |
| 08 | 809 | 813 | 818 | 823 | 828 | 832 | 837 | 842 | 847 | 852 | |
| 09 | 856 | 861 | 866 | 871 | 875 | 880 | 885 | 890 | 895 | 899 | |
| 910 | 904 | 909 | 914 | 918 | 923 | 928 | 933 | 938 | 942 | 947 | |
| 11 | 952 | 957 | 961 | 966 | 971 | 976 | 980 | 985 | 990 | 995 | |
| 12 | 95 999 | *004 | *009 | *014 | *019 | *023 | *028 | *033 | *038 | *042 | |
| 13 | 96 047 | 052 | 057 | 061 | 066 | 071 | 076 | 080 | 085 | 090 | 5<br>1 0.5<br>2 1.0<br>3 1.5<br>4 2.0<br>5 2.5<br>6 3.0<br>7 3.5<br>8 4.0<br>9 4.5 |
| 14 | 095 | 099 | 104 | 109 | 114 | 118 | 123 | 128 | 133 | 137 | |
| 15 | 142 | 147 | 152 | 156 | 161 | 166 | 171 | 175 | 180 | 185 | |
| 16 | 190 | 194 | 199 | 204 | 209 | 213 | 218 | 223 | 227 | 232 | |
| 17 | 237 | 242 | 246 | 251 | 256 | 261 | 265 | 270 | 275 | 280 | |
| 18 | 284 | 289 | 294 | 298 | 303 | 308 | 313 | 317 | 322 | 327 | |
| 19 | 332 | 336 | 341 | 346 | 350 | 355 | 360 | 365 | 369 | 374 | |
| 920 | 379 | 384 | 388 | 393 | 398 | 402 | 407 | 412 | 417 | 421 | |
| 21 | 426 | 431 | 435 | 440 | 445 | 450 | 454 | 459 | 464 | 468 | |
| 22 | 473 | 478 | 483 | 487 | 492 | 497 | 501 | 506 | 511 | 515 | |
| 23 | 520 | 525 | 530 | 534 | 539 | 544 | 548 | 553 | 558 | 562 | |
| 24 | 567 | 572 | 577 | 581 | 586 | 591 | 595 | 600 | 605 | 609 | |
| 25 | 614 | 619 | 624 | 628 | 633 | 638 | 642 | 647 | 652 | 656 | |
| 26 | 661 | 666 | 670 | 675 | 680 | 685 | 689 | 694 | 699 | 703 | |
| 27 | 708 | 713 | 717 | 722 | 727 | 731 | 736 | 741 | 745 | 750 | |
| 28 | 755 | 759 | 764 | 769 | 774 | 778 | 783 | 788 | 792 | 797 | |
| 29 | 802 | 806 | 811 | 816 | 820 | 825 | 830 | 834 | 839 | 844 | |
| 930 | 848 | 853 | 858 | 862 | 867 | 872 | 876 | 881 | 886 | 890 | |
| 31 | 895 | 900 | 904 | 909 | 914 | 918 | 923 | 928 | 932 | 937 | |
| 32 | 942 | 946 | 951 | 956 | 960 | 965 | 970 | 974 | 979 | 984 | |
| 33 | 96 988 | 993 | 997 | *002 | *007 | *011 | *016 | *021 | *025 | *030 | 4<br>1 0.4<br>2 0.8<br>3 1.2<br>4 1.6<br>5 2.0<br>6 2.4<br>7 2.8<br>8 3.2<br>9 3.6 |
| 34 | 97 035 | 039 | 044 | 049 | 053 | 058 | 063 | 067 | 072 | 077 | |
| 35 | 081 | 086 | 090 | 095 | 100 | 104 | 109 | 114 | 118 | 123 | |
| 36 | 128 | 132 | 137 | 142 | 146 | 151 | 155 | 160 | 165 | 169 | |
| 37 | 174 | 179 | 183 | 188 | 192 | 197 | 202 | 206 | 211 | 216 | |
| 38 | 220 | 225 | 230 | 234 | 239 | 245 | 248 | 253 | 257 | 262 | |
| 39 | 267 | 271 | 276 | 280 | 285 | 290 | 294 | 299 | 304 | 308 | |
| 940 | 313 | 317 | 322 | 327 | 331 | 336 | 340 | 345 | 350 | 354 | |
| 41 | 359 | 364 | 368 | 373 | 377 | 382 | 387 | 391 | 396 | 400 | |
| 42 | 405 | 410 | 414 | 419 | 424 | 428 | 433 | 437 | 442 | 447 | |
| 43 | 451 | 456 | 460 | 465 | 470 | 474 | 479 | 483 | 488 | 493 | |
| 44 | 497 | 502 | 506 | 511 | 516 | 520 | 525 | 529 | 534 | 539 | |
| 45 | 543 | 548 | 552 | 557 | 562 | 566 | 571 | 575 | 580 | 585 | |
| 46 | 589 | 594 | 598 | 603 | 607 | 612 | 617 | 621 | 626 | 630 | |
| 47 | 635 | 640 | 644 | 649 | 653 | 658 | 663 | 667 | 672 | 676 | |
| 48 | 681 | 685 | 690 | 695 | 699 | 704 | 708 | 713 | 717 | 722 | |
| 49 | 727 | 731 | 736 | 740 | 745 | 749 | 754 | 759 | 763 | 768 | |
| 950 | 97 772 | 777 | 782 | 786 | 791 | 795 | 800 | 804 | 809 | 813 | |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

TABLE II. COMMON LOGARITHMS OF NUMBERS TO FIVE DECIMAL PLACES

| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 950 | 97 772 | 777 | 782 | 786 | 791 | 795 | 800 | 804 | 809 | 813 |
| | 51 | 818 | 823 | 827 | 832 | 836 | 841 | 845 | 850 | 855 | 859 |
| | 52 | 864 | 868 | 873 | 877 | 882 | 886 | 891 | 896 | 900 | 905 |
| | 53 | 909 | 914 | 918 | 923 | 928 | 932 | 937 | 941 | 946 | 950 |
| | 54 | 97 955 | 959 | 964 | 968 | 973 | 978 | 982 | 987 | 991 | 996 |
| | 55 | 98 000 | 005 | 009 | 014 | 019 | 023 | 028 | 032 | 037 | 041 |
| | 56 | 046 | 050 | 055 | 059 | 064 | 068 | 073 | 078 | 082 | 087 |
| | 57 | 091 | 096 | 100 | 105 | 109 | 114 | 118 | 123 | 127 | 132 |
| | 58 | 137 | 141 | 146 | 150 | 155 | 159 | 164 | 168 | 173 | 177 |
| | 59 | 182 | 186 | 191 | 195 | 200 | 204 | 209 | 214 | 218 | 223 |
| | 960 | 227 | 232 | 236 | 241 | 245 | 250 | 254 | 259 | 263 | 268 |
| | 61 | 272 | 277 | 281 | 286 | 290 | 295 | 299 | 304 | 308 | 313 |
| | 62 | 318 | 322 | 327 | 331 | 336 | 340 | 345 | 349 | 354 | 358 |
| 5 | 63 | 363 | 367 | 372 | 376 | 381 | 385 | 390 | 394 | 399 | 403 |
| 1 0.5 | | | | | | | | | | | |
| 2 1.0 | | | | | | | | | | | |
| 3 1.5 | 64 | 408 | 412 | 417 | 421 | 426 | 430 | 435 | 439 | 444 | 448 |
| 4 2.0 | 65 | 453 | 457 | 462 | 466 | 471 | 475 | 480 | 484 | 489 | 493 |
| 5 2.5 | 66 | 498 | 502 | 507 | 511 | 516 | 520 | 525 | 529 | 534 | 538 |
| 6 3.0 | | | | | | | | | | | |
| 7 3.5 | 67 | 543 | 547 | 552 | 556 | 561 | 565 | 570 | 574 | 579 | 583 |
| 8 4.0 | 68 | 588 | 592 | 597 | 601 | 605 | 610 | 614 | 619 | 623 | 628 |
| 9 4.5 | 69 | 632 | 637 | 641 | 646 | 650 | 655 | 659 | 664 | 668 | 673 |
| | 970 | 677 | 682 | 686 | 691 | 695 | 700 | 704 | 709 | 713 | 717 |
| | 71 | 722 | 726 | 731 | 735 | 740 | 744 | 749 | 753 | 758 | 762 |
| | 72 | 767 | 771 | 776 | 780 | 784 | 789 | 793 | 798 | 802 | 807 |
| | 73 | 811 | 816 | 820 | 825 | 829 | 834 | 838 | 843 | 847 | 851 |
| | 74 | 856 | 860 | 865 | 869 | 874 | 878 | 883 | 887 | 892 | 896 |
| | 75 | 900 | 905 | 909 | 914 | 918 | 923 | 927 | 932 | 936 | 941 |
| | 76 | 945 | 949 | 954 | 958 | 963 | 967 | 972 | 976 | 981 | 985 |
| | 77 | 98 989 | 994 | 998 | *003 | *007 | *012 | *016 | *021 | *025 | *029 |
| | 78 | 99 034 | 038 | 043 | 047 | 052 | 056 | 061 | 065 | 069 | 074 |
| | 79 | 078 | 083 | 087 | 092 | 096 | 100 | 105 | 109 | 114 | 118 |
| | 980 | 123 | 127 | 131 | 136 | 140 | 145 | 149 | 154 | 158 | 162 |
| | 81 | 167 | 171 | 176 | 180 | 185 | 189 | 193 | 198 | 202 | 207 |
| 4 | 82 | 211 | 216 | 220 | 224 | 229 | 233 | 238 | 242 | 247 | 251 |
| 1 0.4 | 83 | 255 | 260 | 264 | 269 | 273 | 277 | 282 | 286 | 291 | 295 |
| 2 0.8 | | | | | | | | | | | |
| 3 1.2 | | | | | | | | | | | |
| 4 1.6 | 84 | 300 | 304 | 308 | 313 | 317 | 322 | 326 | 330 | 335 | 339 |
| 5 2.0 | 85 | 344 | 348 | 352 | 357 | 361 | 366 | 370 | 374 | 379 | 383 |
| 6 2.4 | 86 | 388 | 392 | 396 | 401 | 405 | 410 | 414 | 419 | 423 | 427 |
| 7 2.8 | | | | | | | | | | | |
| 8 3.2 | 87 | 432 | 436 | 441 | 445 | 449 | 454 | 458 | 463 | 467 | 471 |
| 9 3.6 | 88 | 476 | 480 | 484 | 489 | 493 | 498 | 502 | 506 | 511 | 515 |
| | 89 | 520 | 524 | 528 | 533 | 537 | 542 | 546 | 550 | 555 | 559 |
| | 990 | 564 | 568 | 572 | 577 | 581 | 585 | 590 | 594 | 599 | 603 |
| | 91 | 607 | 612 | 616 | 621 | 625 | 629 | 634 | 638 | 642 | 647 |
| | 92 | 651 | 656 | 660 | 664 | 669 | 673 | 677 | 682 | 686 | 691 |
| | 93 | 695 | 699 | 704 | 708 | 712 | 717 | 721 | 726 | 730 | 734 |
| | 94 | 739 | 743 | 747 | 752 | 756 | 760 | 765 | 769 | 774 | 778 |
| | 95 | 782 | 787 | 791 | 795 | 800 | 804 | 808 | 813 | 817 | 822 |
| | 96 | 826 | 830 | 835 | 839 | 843 | 848 | 852 | 856 | 861 | 865 |
| | 97 | 870 | 874 | 878 | 883 | 887 | 891 | 896 | 900 | 904 | 909 |
| | 98 | 913 | 917 | 922 | 926 | 930 | 935 | 939 | 944 | 948 | 952 |
| | 99 | 99 957 | 961 | 965 | 970 | 974 | 978 | 983 | 987 | 991 | 996 |
| | 1000 | 00 000 | 004 | 009 | 013 | 017 | 022 | 026 | 030 | 035 | 039 |
| Prop. Parts | N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

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